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# Geometric Representations Of Functions Of a Complex Variable

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# GEOMETRIC REPRESENTATION

OF

# FUNCTIONS OF A COMPLEX VARIABLE

BY

EDWIN RAYMOND SMITH

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THESIS FOR DEGREE OF BACHELOR OF ARTS

IN MATHEMATICS

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COLLEGE OF LITERATURE AND ARTS

UNIVERSITY OF ILLINOIS

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OF Bachelor of Arts.

E. W. Chittenden

HEAD OF DEPARTMENT OF Mathematics.

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# Preface.

It is the purpose of this paper to discuss the ways in which it is possible to give a geometric interpretation to functions of a complex variable, and to apply the principles which may be developed in the representation of a few typical examples. Special attention will be given to those functions which are represented by the models possessed by the mathematical department of the University of Illinois. These functions include

$w = \sqrt{z}$, $w = z^{\frac{1}{n}}$, and $w = \sqrt[3]{\frac{z-a}{z-b}} + \sqrt{z-c}$,

of the corresponding Riemann Surfaces. They also include among those represented by Isothermic Surfaces,

$w^2 = z^2 - 1$, $w^2 = z^4 - 1$, and $w^4 = 1 - z^2$,

as types of algebraic functions.

$w = \frac{1}{z}$, $w = \frac{1}{2\varepsilon} \log \frac{z-\varepsilon}{z+\varepsilon}$, and $6w = \varepsilon^{\frac{1}{6z}}$ as types of logarithmic functions. The latter surfaces are included in Series XIV of Bill's Models.

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# Chapter I.

## Introduction

1. Complex number, complex variable, and function of a complex variable. A complex number is a quantity consisting of a real and an imaginary part connected by addition. Such numbers can always be reduced to the form,

$$z = x + iy,$$

in which $x$ is the real part and $iy$ the imaginary part. Should $y$ be equal to $0$ there is obtained a real number, or should $x$ equal $0$, the pure imaginary number.

A complex variable arises when one or both of the real numbers, $x$ or $y$, becomes a variable independently of the other.

When a complex variable in the

form of

$$w = u + iv$$

is obtained by subjecting another complex variable,

$$z = x + iy,$$

to mathematical operations, and if there exists such a relation between $w$ and $z$ that the partial differential equations[x]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied, then $w$ is called a function of the complex variable $z$.

2. Single- and Multi-valued functions. When for a complex variable $z$ another complex variable $w$ can be assigned, so that when $z$ is given a particular value, $w$ has one and only one corresponding value, $w$ is called a single-valued function of $z$.

[x] Durège, Elements of the Theory of Functions page 21.

When for a given $z$, two or more values of $w$ exist, $w$ is a <u>multi-valued function</u> of $z$.

3. <u>Argand diagram.</u> A complex variable in the form,

$$z = x + iy,$$

depends on two independent real variables $x$ and $y$, each of which may assume a singly infinite series of values. For this reason it is impossible to represent a complex number by points along a single line. Consequently, some kind of a surface must be chosen as the field in which the points are located. When the plane is used the figure formed by mapping a function of a complex variable is referred to as <u>Argand's diagram</u>, or sometimes as the Gauss or the Wessel plane. In the Argand diagram the points are located by reference to a fixed system of right-

angled coordinates similar to those of Descartes for real variables. Along one axis is determined the real part of the variable, and along the other the imaginary part. The axes are, therefore, designated as the axis of real variables, and the axis of imaginary variables. The point z is that point in the Argand plane which has for its coordinates the lengths of the axes which correspond to the real and imaginary parts of z. In

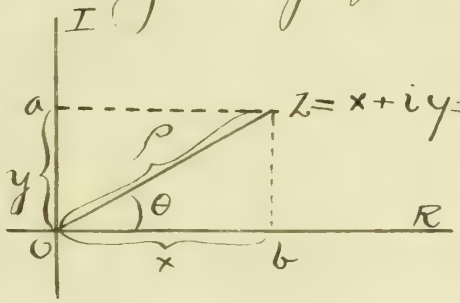


Fig 1.

figure 1, O is the origin, $\overline{OR}$ the axes of real variables, $\overline{OI}$ the axis of imaginary variables. Let z be any point in the plane, then the distance $\overline{Ob}$ corresponds to x and the distance $\overline{Oa}$ to y. In this manner the position of

$z$ is uniquely determined.

Instead of using the rectangular coordinates $x$ and $y$, we can accomplish the same result by means of polar coordinates. We can put

$$x = r\cos\theta, \text{ and}$$
$$y = r\sin\theta,$$

and obtain

$$z = r(\cos\theta + i\sin\theta).$$

When the complex variable is in this form $r$ is always taken positively and is called the <u>modulus</u> of the complex variable $z$. Geometrically it is the distance of the point from the origin or the length $Oz$ (figure 1). The angle $\theta$ is called the <u>amplitude</u>, or the <u>argument</u> of $z$, and it shows the direction in which the point lies from the origin. In figure 1, it is the angle $bOz$.

4. Argand diagram adapted to the

representation of a single-valued function.

From the above, it follows that there are two different ways by which a point can be located in the Argand plane, that is, by either rectangular or polar coordinates. So long as only single-valued functions are considered, the scheme serves the purpose for any ordinary geometric representation. Two planes, a z-plane and a w plane, can be used, and for a definite value of z in the z-plane, there is one and only one corresponding value of w in the w-plane. If there be a path leading from $z_0$ to $z_1$, there will be a path leading from the initial value of the function $w_0$ to the corresponding final value $w_1$. Take for example the linear function,

$$w = f(z) = az + b,$$

in which w and z are complex variables,

and $a$ and $b$ are complex constants.
Let

$$a = 2+i, \text{ and}$$

$$b = 1+2i.$$

Suppose that in the $z$-plane, $z$ starting from the origin moves by a straight line to the point where

$$z = 3+2i.$$

At the same time $w$ would move from its

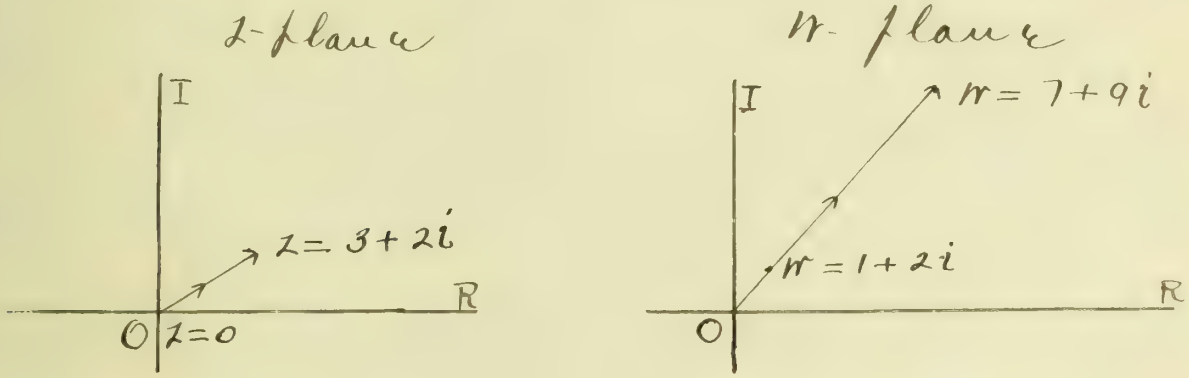


Fig. 2.

initial point,

$$w = 1+2i,$$

to its final position in the plane,

$$w = 7+9i.$$

No other point in the $w$-plane corresponds

to the final value of $z$, and there is no other point in the $z$-plane which corresponds to the final value $w$. There is, therefore, a one to one correspondence between the points of the two planes.

It is often desirable to consider the $z$-plane and the $w$-plane as coincident, and with the same axes. In such cases $z$ and $w$, or $f(z)$ are different places in the same Argand plane.

5. Graphical representation of multi-valued functions. A difficulty arises when it is desired to represent a multi-valued function by means of the Argand-Gauss diagram. Since for each value of $z$ there corresponds more than one value of $w$, it follows that for each point in the $z$-plane there may be two or more corresponding points in the $w$-plane. A path which leads from $z_0$

to $z_0$ in the $z$-plane, may in the $w$-plane lead from $w_0$ to entirely different values of $w$. It follows that such a representation would result, quite often, in indefiniteness and confusion.

The two valued function,

$$w = f(z) = \sqrt{z},$$

will show how this difficulty arises. When a definite value $z_0$ is given to $z$, $w$ will have for its values $+\sqrt{z_0}$ and $-\sqrt{z_0}$. So when a path moves from the the origin to $z_0$ in the $z$-plane, it is not known whether the path in the $w$-plane will move from the origin to the point where $w$ has the value $+\sqrt{z_0}$, or to the point where $w$ has the value $-\sqrt{z_0}$. The final value of the function is not completely determined.

For further consideration of this function it will be best to change it into

the polar form, in which

$$z = \rho(\cos\theta + i\sin\theta), \text{ and}$$

$$w = r(\cos\psi + i\sin\psi).$$

Then on making these substitutions,

$$r(\cos\psi + i\sin\psi) = [\rho(\cos\theta + i\sin\theta)]^{\frac{1}{2}}, \text{ or}$$

$$r^2(\cos 2\psi + i\sin 2\psi) = \rho(\cos\theta + i\sin\theta).$$

It follows then that,

$$\rho = r^2, \text{ and}$$

$$\theta = 2\psi.$$

From these two last equations it can be seen that when $z$ makes a closed circuit around the origin and takes values from 0 to $2\pi$, $w$ will have made a circuit around the origin from 0 to $\pi$. When $z$ makes

$z$-plane $\qquad$ $w$-plane

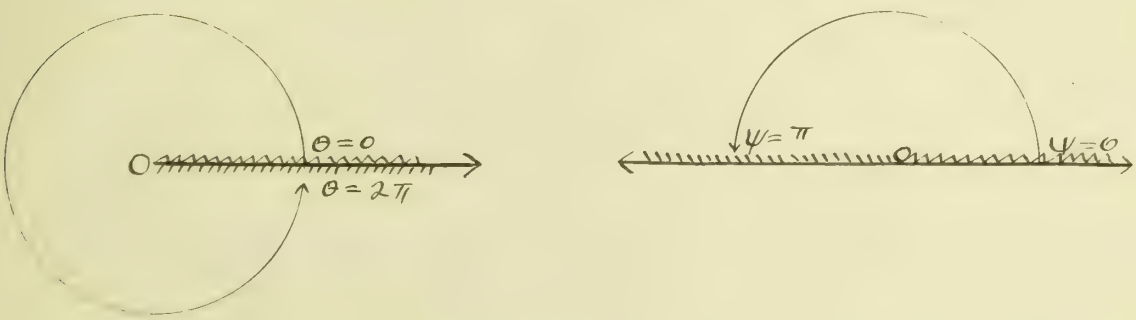


Fig. 3

a second circuit around the origin, that is, takes values from $2\pi$ to $4\pi$, $w$ will go from $\pi$ to $2\pi$, thus completing its circuit around the origin. From this it follows that for each point in the $z$-plane there corresponds two distinct points in the $w$-plane to which different values of the function are attached.

To do away with this confusion, which results from attempts to represent multi-valued functions in the Argand plane, different methods of representation have been devised in which one and only one value of $w$ corresponds to a given value of $z$ in the $z$-plane. Riemann used a many sheeted $z$-surface and gave to each sheet a particular set of function values. Darboux and others separated the function into its real and imaginary parts and gave to each a separate representation. Either of

these means gives to the multi-valued function a definite and complete representation, and makes it possible to treat the function as though it were single-valued. We will first turn our attention to the Riemann Surface.

# Chapter II

## Riemann Surfaces

6. The function $w = \sqrt{z}$. As has already been shown the function,

$$w = \sqrt{z},$$

is a two-valued function, and as such cannot be uniquely represented on a single plane by means of the Argand diagram. For this function Riemann used two thin sheets lying infinitely near each other. For convenience the $z$-surface may be considered as two horizontal sheets, one directly above the other. Since for a given value of $z$, $w$ has two distinct values, the points which correspond to them may be taken at the places on the sheets where they are cut by a vertical straight line. Should $z$ be equal

to $z_0$, the place on the upper sheet may take the value $+\sqrt{z_0}$, and the place directly below on the lower sheet the value $-\sqrt{z_0}$. By this scheme it is evident that each place on the two sheets has one and only one value of w attached to it. There are, however, certain values of z for which the function values are identical. Zero and infinity are such values for the particular function under consideration. It follows, then, that at 0 and $\infty$, the points at which the vertical line cuts the two sheets are identical, that is, the two sheets come together at the points. It may be said, then, that the surface consists of two sheets lying infinitely near each other and hanging together at 0 and $\infty$.

There must, however, be further connections between the two sheets as is shown by the following considerations. In the Argand diagram

a path going once around the origin in the $z$-plane changes the value of the function from $+\sqrt{z}$ to $-\sqrt{z}$. That is, if $z_0$ and $w_0$ are points on the positive real axes of their respective planes, and a path, which is closed and which passes once around the origin, restores the initial value of $z$, but leaves $w$ at a point on the negative real axis. It follows from this that if on the Riemann surface, a path starts from a place on the upper sheet, to which a positive value of the function is attached $+\sqrt{z_0}$, and passes once around the origin, it takes on the negative value $-\sqrt{z_0}$, which belongs to the place on the lower sheet directly below the starting place. Consequently at some point in the circuit the path changes from the upper to the lower sheet. To make this change possible there must be a bridge reaching from the origin

to infinity over which it is possible for the path to go from the upper to the lower sheet. Moreover, if a path is described twice around the origin in the same direction the original value of the function is restored. In the double circuit the values change from $+\sqrt{z}$ to $-\sqrt{z}$ and from $-\sqrt{z}$ back to $+\sqrt{z}$, and in the surface the path goes from the upper sheet into the lower sheet and from the lower sheet back to the upper sheet. There must, therefore, be a second bridge which provides a passage from the lower sheet back into the upper one. It is now seen that the surface not only hangs together at $0$ and $\infty$, but has a double bridge, which does not intersect itself, reaching from $0$ to $\infty$. The points of connection $0$ and $\infty$ are called <u>branch-points</u> and the double bridge is known as a <u>branch cut</u>.

That portion of the surface which lies around the origin is shown in figure 4. The branch point at O is the point O. The

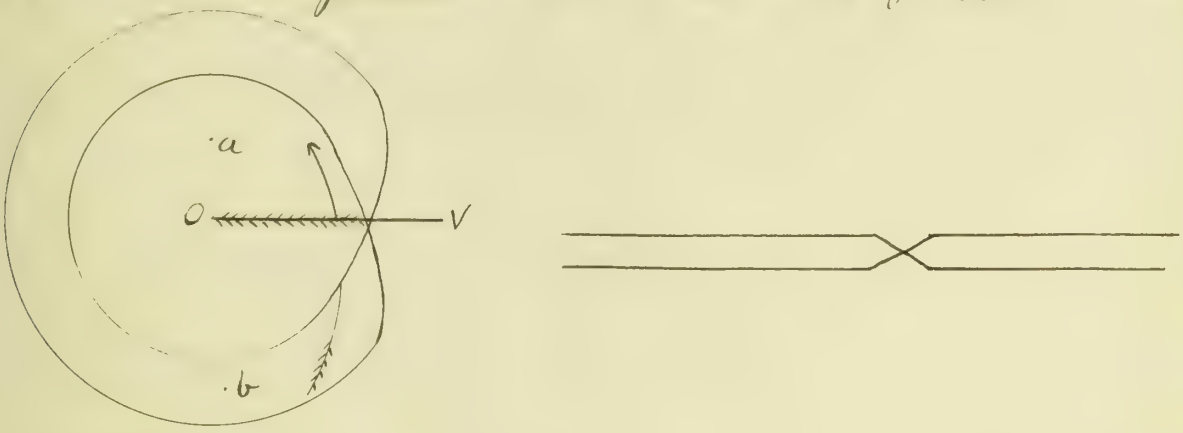


Fig. 4

branch cut is along the line $\overline{OV}$. The point a is in the upper sheet and b in the lower one. The lines shown on the right are formed by cutting the surface by a plane perpendicular to the branch cut. The two sheets lie infinitely near each other and are extended indefinitely in all directions. When a path makes a circuit around the origin it changes from one sheet to the other at the branch cut $\overline{OV}$, as indicated by the arrow. It is easy to see that a closed

curve around the origin must make two circuits and consequently crosses the branch cut twice.

7. Branch cuts. So far the double bridge has been defined only by its extremities, the branch points, and that it must not intersect itself. Nothing has been said as to its position in regard to the planes. It will now be shown that this need not be taken into consideration.

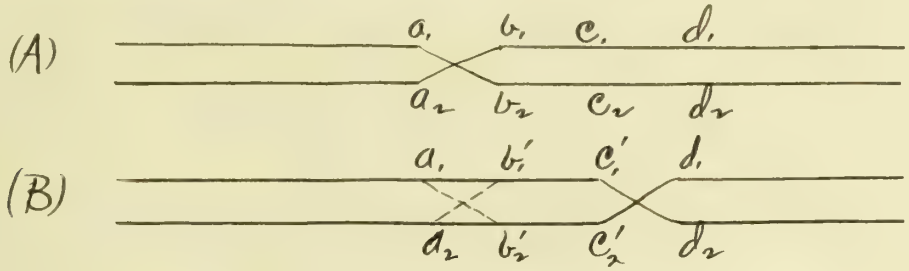


Fig. 5

In figure 5 let (A) be a section of the surface made by a plane perpendicular to the branch cut. It can be seen that $a_1$ and $b_2$ or $a_2$ and $b_1$, are consecutive values of the variable, on different sides of the branch cut.

On the other hand $a_1$ and $b_1$, or $a_2$ and $b_2$, are not consecutive values. Thus if the function value at $a$, be $+\sqrt{a}$, and at $a_2$, $-\sqrt{a}$, the principle of continuity requires that the value at $b_2$ shall be $+\sqrt{b}$, and that of $b_1$, $-\sqrt{b}$, $c_2$, $d_2$, etc. will take positive values and $c_1$, $d_1$, etc. negative values. Suppose now that the branch cut be shifted so that it occupies the position indicated in (B). A comparison of the two conditions shows that the points which have changed sheets have carried their values over with them. Thus $b_1$ and $c_1$ retaining their original values go over into the lower sheet to the places $b_2'$ and $c_2'$. At the same time $b_2$ and $c_2$ go into the upper sheet to the places $b_1'$ and $c_1'$. Thus it is seen that a shifting of the branch cut changes that part of the upper sheet between the original and final positions of the branch cut into

part of the lower sheet, and the corresponding part of the lower sheet goes over into the upper one. Since this change does not disturb the value of the function which is attached to a particular place on the sheets, it follows that it makes no difference as to the position of the branch cut, so long as it reaches from $c$ to $\infty$. A similar law holds for the general Riemann surface.

1. Closed path restores the initial value of the function. It remains yet to show

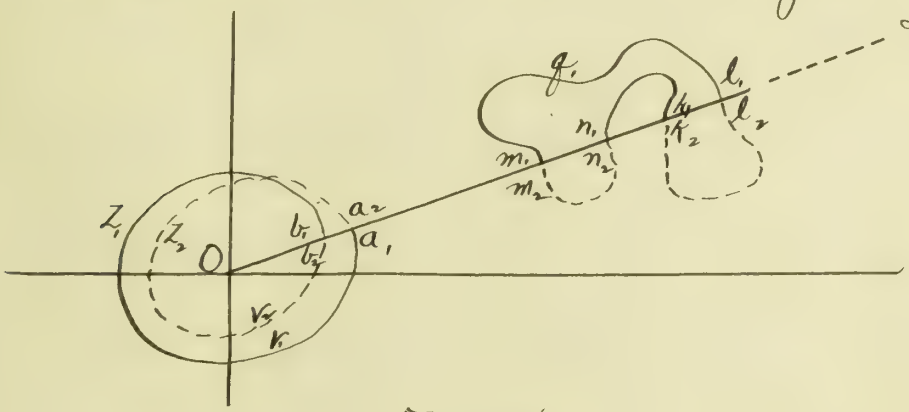


Fig. 6

that a path which restores the initial value of the function gives a closed path on the Riemann surface. For convenience the places on

the upper sheet will be designated by the subscript one, and those in the lower sheet by the subscript two. Thus $z_1$ and $z_2$ lie in the places at which a vertical line cuts the horizontal sheets. Now, a closed path which starts at $z_1$ must end at $z_1$, and not at $z_2$. A path including the branch point o, which starts at $p_1$ (figure 6) and ends at $p_1$, passes through $r_1$ and changes from $a_1$ to $a_2$ at the branch cut which reaches from o to $\infty$, and after passing through $p_2$ and $r_2$ changes from $b_2$ to $b_1$ at the branch cut and ends at $p_1$. Such a path which restores the initial value of the function is seen to be a closed curve.

Again a path which does not include the origin, starting at $q_1$ and ending at the same value $q_1$, passes at m into the lower sheet, at n into the upper sheet, at k into the lower sheet, and at l back into

the upper sheet. The path, $q, m, m_2 n_2 n, k, k_2 l_2 l, q$, which restores the initial value of the function is evidently a closed curve.

Any path which does not pass through the branch cut remains at all times in the same sheet and must be closed when the initial value of the function is restored. By tracing out other curves it can be seen that on a two sheeted Riemann surface any path which restores the initial value of the function meets the branch cut an even number of times and is therefore a closed curve. In this proposition 0 is considered as an even number.

We have now succeeded in producing a surface on which the two-valued function,

$$w = f(z) = \sqrt{z},$$

can be uniquely represented and on which it obeys the same laws as a single valued function in the Argand plane.

9. The two-valued function $w = \sqrt{(z-a)(z-b)}$.

Let us now consider the more general two valued function,

$$w = f(z) = \sqrt{(z-a)(z-b)}.$$

In the first place it is two valued and will, therefore, require a two sheeted Riemann surface, in order that each value of $w$ may correspond to a particular place on the sheets. At the points $a$ and $b$ the function is equal to zero, and consequently the function will have equal values at those places. Hence, the branch points of the function are at the points $a$ and $b$. Now suppose that a double bridge extends from $a$ to $b$. Then, the surface would consist of two sheets hanging together at $a$ and $b$, and passing into each other along a line connecting these points. At all other places the sheets would be unconnected and any path which changes

from one sheet to another must meet this bridge. A closed circuit which passes once around both branch points would either not meet this bridge at all or else cross it an even number of times, thus restoring the initial value of the function. A closed circuit which passes twice around either branch point meets the double bridge two times, and again the initial value of w is restored. Finally, any closed circuit which crosses the branch cut an even number of times, but which does not enclose a branch point, restores the value of w. These considerations show that the double bridge connecting a and b gives all necessary opportunities for a path, which is defined by the function, to change from one sheet to the other. It follows then that the surface consists of two sheets extending indefinitely in all directions, with branch points at a and b,

and with a double bridge, or branch cut, connecting the branch points. Figure 7 shows

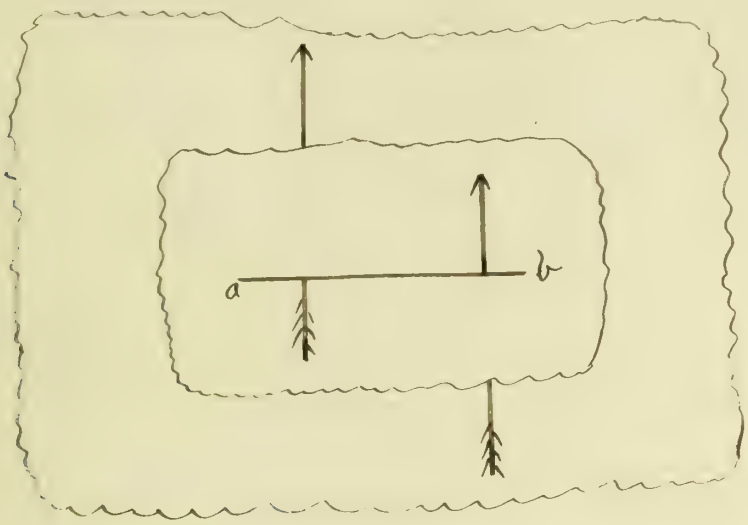


Fig. 7

a portion of such a surface. The branch cut is drawn as a straight line reaching from a to b. There are, however, other branch cuts which satisfy the conditions as will be shown later in connection with the Riemann sphere.

10. Riemann's Sphere. Riemann's sphere is an adaptation of the Neumann sphere[x] which is often used in connection with the Argand diagram. It is a means by which

(x) Harkness and Morley, Treatise on Function Theory, page 27.

the surface consisting of infinite sheets is transformed into the closed surface of the sphere. In this manner it will be seen that it is possible to speak of all the points of the plane which lie at infinity, as a particular place on the sphere.

Let a sphere of arbitrary radius be tangent to the plane at the origin. For convenience consider the point of tangency as the south pole and the point diametrically opposite as the north pole. Then any point L of the plane can be projected on the surface of the sphere by drawing a straight line from the north pole of the sphere to the point L, and the point at which this line cuts the surface of the sphere is the required point. The origin becomes the south pole of the sphere, and all points at infinity become the north pole. The points in the finite region

of the plane complete the surface of the sphere. It follows that there is a one to one correspondence between the points in the plane and the points in the sphere's surface, that is, for each point in the plane there is one and only one corresponding point in the sphere and for each point on the sphere there is one and only one point in the plane.

Since the Riemann surfaces which represent the functions,

$$w = \sqrt{z}, \text{ and}$$

$$w = \sqrt{(z-a)(z-b)},$$

consist of two infinite sheets lying infinitely near each other, the corresponding spherical surfaces consist of two concentric spheres whose surfaces lie infinitely near each other, connected together at the branch points, and with the outer and inner surfaces connected together along the branch cuts.

In the first function the branch points are 0 and $\infty$, therefore, the spheres touch each other at the north and south poles. The branch cut, which in the plane has been defined as reaching from 0 to $\infty$ regardless of its direction, on the spherical surface becomes any line which connects the two poles. The simplest arrangement is that the branch cut shall be one half of any great circle which reaches from pole to pole.

It is easy to see that the second function,

$$w = \sqrt{(z-a)(z-b)},$$

is only a more general type which includes such functions as

$$w = \sqrt{z},$$

and many others which have two branch points and are two-valued. In the more general case, instead of having the branch

points at the particular plane c and x, they are the arbitrarily chosen points a and b, which may take any values whatever. The branch cut reaches from a to b along any line in the surface which does not intersect itself. It may be supposed that this line passes through the north pole of the sphere. It is now possible by an examination of the plane to find the lines whose projections on the sphere give the line reaching from a to b through x. Lines in the plane which reach from a and b to x give the required projections. Hence, instead of taking a single branch cut in the plane, as in section 9, two branch cuts reaching from the branch points to infinity can answer the purpose equally well. Similarly they can all reach to any point in the finite regions of the plane.

which is not itself a branch point.

11. Function $w = \sqrt{z(1-z)(a-z)(b-c)(c-z)}$.

As an example, to show the ways in which it is possible to connect the sheets of a Riemann surface, we will discuss the two-valued function

$$w = f(z) - \sqrt{z(1-z)(a-z)(b-z)(c-z)}$$

It is at once evident that the values of $z$ which locate branch points are $0, 1, a, b, c$, and $\infty$. The surface is two sheeted with connections at these points. From the principles which have already been developed in the discussion of the functions which have been taken up, it follows that when a closed curve is obtained, the path either does not enclose a branch point, or else encircles a branch point an even number of times. Since the essential principle is that the path shall meet a branch cut an even number of times, it follows that a closed curve can be obtained on a two-sheeted

Riemann surface by going around an even number of branch points. From this we can deduce the general law that a closed curve can be obtained in the n sheeted surface by going around n branch points. When a branch point is encircled more than once it should be counted as many times as it is encircled. On the other hand should a curve in a two-sheeted surface enclose an odd number of branch points the initial value is restored but with a change of sign. For the surface under consideration, it can be seen that branch cuts along straight lines reaching from 0, 1, a, b, and c to ∞ satisfy the requirements, but a simpler arrangement is to join 0 to 1, a to b, and c to ∞. Figure 6 shows such an arrangement. These branch cuts must not intersect themselves or each other. The figure also shows that paths

which includes an even number of branch points are closed on the Riemann surface, while the curve $q_1 S R q_2$, which includes

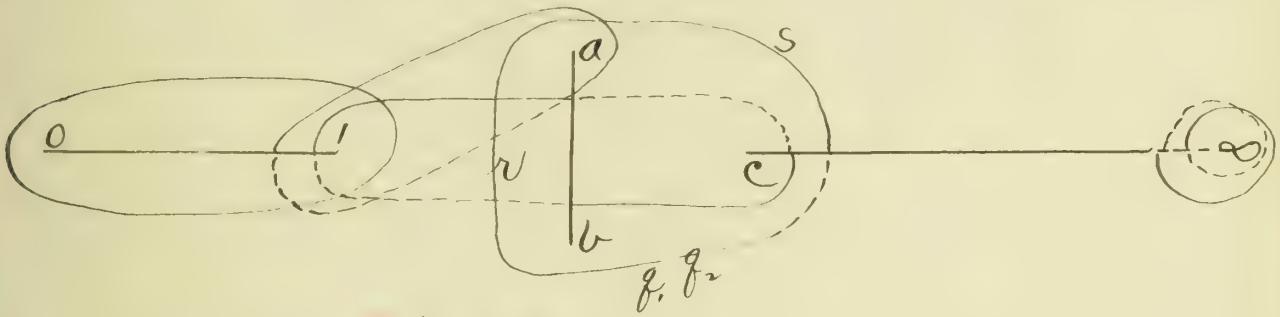


Fig 8.

an odd number of branch points, begins in the upper sheet and ends in the lower.

It is possible to find several other different ways of connecting the branch points with branch cuts and still have the conditions such that when a curve passes around an even number of branch points the initial value of the function is restored and the curve is closed. This, however, will serve for our present purpose.

12. Function $w = z^{\frac{1}{n}}$. The function,

$$w = f(z) = z^{\frac{1}{n}},$$

is an $n$-valued function such to a given $z$ there corresponds $n$ values of $w$. It will be necessary for this function to have an $n$-sheeted Riemann surface, in order that one value of $w$ may correspond to one place on the sheets, and only one. Let the $n$ values of $w$ for a given value of $z$ be,

$$w_1 = z^{\frac{1}{n}},\ w_2 = \alpha w_1,\ w_3 = \alpha^2 w_1,\ \ldots,\ w_n = \alpha^{n-1} w_1,$$

where,

$$\alpha = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}.$$

It is evident that for zero and infinity these values of the functions will be equal. Hence, the branch points will be $0$ and $\infty$. It makes no difference as to how the sheets of the $z$-plane are numbered, but it will be convenient to let $w_1, w_2, \ldots w_n$ belong initially to the 1st, 2nd, 3rd, ..., $n^{th}$ sheets respectively. In the Argand diagram one circuit around the origin changes $z^{\frac{1}{n}}$ into $\alpha z^{\frac{1}{n}}$, that is,

$w_1$ is changed into $w_2$, and so on making a second circuit $w_2$ is changed into $w_3$. Finally, the $n$th circuit changes $w_n$ into $w_1$ which is the initial value of the function.

Using the notation which has been used in previous sections $z_1, z_2, z_3, \ldots, z_n$ are $n$ places where a vertical line cuts all of the sheets, and to all of which $z$ is attached. The first circuit must lead from $z_1$ to $z_2$, the second from $z_2$ to $z_3$, and so on until in the $n$th circuit the path leads from $z_n$ back to $z_1$, and the curve is closed by a restoration of the initial value $z_1$. It will be necessary to devise a branch cut which will give the necessary connection between the sheets to allow of such changes from one sheet to another. Figure 9 shows a section of the surface made at right angles to a branch cut having the necessary

connections, as viewed from the origin. On meeting such a branch cut the path changes

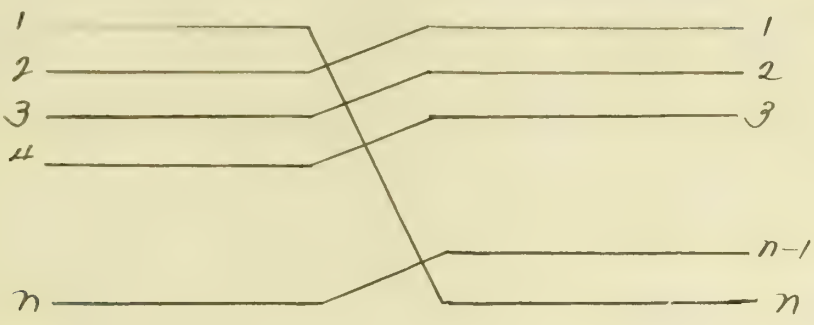


Fig 9.

from one sheet to another. Thus, a path which starts in sheet 1 on making a circuit around the origin passes into sheet 2. A second circuit in the same direction brings the path into the sheet 3, and so the changes continue until in the nth circuit the path goes from sheet n back into sheet 1. In this way the initial value of the function is restored and the path forms a closed circuit. A closed path which does not include the origin must, either not meet a branch cut at all, or else cross it

the same number of lines in both directions.

13. Branch points for the general algebraic function. After having studied a few typical examples of the Riemann surfaces, we are now ready to take up the discussion of the n-sheeted surface for the general algebraic function

$$f(w, z) = 0$$

In the first place, the number of sheets will be determined by the number of values which the function takes for a given $z$. A separate sheet will be required for each of the different values. In general the value $n$ corresponds to the degree of the function in $w$, when the equation is free from radicals and in its simplest form. As the second step it will be necessary to locate the branch points, and find out the behavior of function in their neighborhood. As has already been pointed out, a

branch point is a point in the surface at which two or more values of the function are equal, and such that there is a change of function values when a path describes a small circuit around it. That the function values must be equal is evident from the fact that at the branch points the n sheets must be connected, that is, have a point in common. That the function values must change does not imply that all of them must change during a particular circuit. It is only necessary for this change to occur in the case of some one of the values under consideration. The nature of the change may be such that when a circuit is made round a branch point only a part of the values may change, while the others remain unchanged.

In order to locate a branch point, it

is necessary to find those places at which the multi-valued function has equal values. By the theory of functions this condition can be determined by eliminating $w$ from the two equations,[x]

$$f(w, z) = 0$$

$$\frac{\partial f(w, z)}{\partial w} = 0$$

The equation in $z$, which is thus formed, has among its roots those values of $z$ which give the branch points. In order to determine which of the roots are to be chosen, it is necessary to pass a small circuit around the point, and see if the function changes from one sheet into

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[x] This follows from the fact that a multiple root of the order $m$ is also a multiple root of the order $m-1$ of the first derived equation.

Forsyth, Theory of Functions, Second Edition pp. 181.

Burnside and Panton, Theory of Equations, pp. 158.

another, or changes its value during the circuit. These values may in the simplest case change consecutively or generally in more complicated cyclic systems.

14. Cyclic systems of branch points. The way in which a path, which goes around a branch point, changes from one sheet into another along the branch cuts determines what is known as the cyclic systems of the branch point. A path, while making such a circuit, does not necessarily pass into the sheet which is adjacent to the one in which the initial point lies. It may, perhaps, pass from the first sheet into the third instead of going directly into the second, or it may go from the fourth back into the first, etc. The way in which the sheets are connected in the Riemann surface, along a branch cut depends on the character of the function. To

find out the way in which the sheets permute into one another is to determine the cyclic systems of the branch points, which lie at the extremities of the branch cut along which the permutations take place. To do this start from a non-critical point, $z_0$, on the surface and pass a circuit around the branch point, returning again to $z_0$, noting the change in the value of the function. Such a circuit on the Riemann surface must cross a branch cut, since a branch cut can always be extended from a branch point to infinity. Make similar circuits around all the other branch points and continue the circuits until the original value, $z_0$, is restored and the path closed. Having in this way found out the manner in which the values change, the sheets must be connected along the different branch cuts in such a way as

to have consecutive values in a continuous path in the sheets.

This process is very well shown in the construction of the surface for the function

$$w = \sqrt[3]{\frac{z-a}{z-b}} + \sqrt{z-c}.$$

This is a root of the equation,

$$(z-b)^2 w^6 - 3(z-b)^2(z-c)^2 w^4 - 2(z-a)(z-b)\, w^3 + 3(z-b)^2(z-c)^2 w^2 - 6(z-a)(z-b)(z-c)\, w + (z-a)^2 - (z-b)^2(z-c)^3 = 0,$$

which is an equation of the sixth degree in $w$. Consequently, there must be six sheets in the Riemann surface. The branch points are at $a$, $b$, $c$, and $\infty$.

In determining the cyclic systems of the branch points, the following substitutions will be helpful.

$$\sqrt[3]{z-a} = t$$

$$\sqrt[3]{z-b} = w$$

$$\sqrt{z-c} = v$$

Let us first consider the path which $z$ makes in passing around $a$. For these circuits $t$ passes into $\alpha t$, $\alpha^2 t$, $t$, etc., while $w$ and $v$ remain unchanged. To begin with, let the values be arranged in the order,

$$w_1 \; w_2 \; w_3 \; w_4 \; w_5 \; w_6.$$

After the first circuit the sequence changes into,

$$w_2 \; w_3 \; w_1 \; w_5 \; w_6 \; w_4.$$

after a second circuit it becomes,

$$w_3 \; w_1 \; w_2 \; w_6 \; w_4 \; w_5.$$

After a third circuit it is,

$$w_1 \; w_2 \; w_3 \; w_4 \; w_5 \; w_6$$

which is the original order. It is seen, that around this branch point only the values $w_1 \; w_2 \; w_3$ permute by themselves, and $w_4 \; w_5 \; w_6$ by themselves.

For circuits around the point $b$, $t$ and $v$ remain unchanged, and $w$ changes

into $\alpha w, \alpha^2 w, w$, etc. Therefore, beginning with,

$$w_1\ w_2\ w_3\ w_4\ w_5\ w_6,$$

as the original sequence, after the first circuit it changes into,

$$w_3\ w_1\ w_2\ w_6\ w_4\ w_5$$

After a second circuit it becomes,

$$w_2\ w_3\ w_1\ w_5\ w_6\ w_4.$$

After a third circuit,

$$w_1\ w_2\ w_3\ w_4\ w_5\ w_6,$$

which is the original order of sequence. Here the same function values permute as for the branch point $a$, but the order of sequence is reversed.

Finally on making circuits around $c$, $t$ and $u$ remain unchanged while $v$ changes into $-v$ and $+v$, etc. Arranging the values in the same sequence as before,

$$w_1\ w_2\ w_3\ w_4\ w_5\ w_6,$$

and passing once around c we obtain the sequence,

$$w_4\ w_5\ w_6\ w_1\ w_2\ w_3,$$

After a second circuit we have,

$$w_1\ w_2\ w_3\ w_4\ w_5\ w_6.$$

Thus the original sequence is restored by making two circuits around c.

From this discussion it can be seen that the function

$$w = \sqrt[3]{\frac{z-a}{z-b}} + \sqrt{z-c}$$

has two branch points around which the three values $w_1\ w_2\ w_3$ permute in cyclic order, but never with one of the three remaining ones. In a similar manner $w_4\ w_5\ w_6$ permute in cyclic order, but never pass into one of the first three values. From this it follows, that the branch cuts going out from a and b need not go to infinity, for a cut connecting the two points will

answer the purpose equally well. This is due to the correspondence of the cyclic systems of the two points. We still have one more branch point C, at which the three pairs of values $w, w_4$, $w_2 w_3$, and $w_5 w_6$, each by itself, permute their values, and at no time does a value from one pair pass into a value found in another pair. Since there is no other point in the finite region of the plane which is similar to C, the branch cut from C must of necessity reach out to infinity.

If we let a closed line include two branch points the result is the same as if a circuit had been made around each point separately, since the two circuits can always be changed into one by simply connecting the terminal point of the first circuit with the initial point of the second one.

In the example which is being considered a circuit around, say a and c, will bring about the same result as if the circuit had gone around a and c separately. In such a circuit both of the branch cuts are crossed. We can therefore, determine the connection between the sheets along the branch cuts, which will permit the consecutive function values to lie in their respective sheets, as z describes a closed path in the direction of increasing angles. The following table shows the changes of values as the circuits go round a and c.

| circuits | round a | round c | round both |
|---|---|---|---|
| 1 | $w_1$ changes into $w_2$ | $w_2$ into $w_5$ | $w_1$ into $w_5$ |
| 2 | $w_5$ " " $w_6$ | $w_6$ " $w_3$ | $w_5$ " $w_3$ |
| 3 | $w_3$ " " $w_1$ | $w_1$ " $w_4$ | $w_3$ " $w_4$ |
| 4 | $w_4$ " " $w_5$ | $w_5$ " $w_2$ | $w_4$ " $w_2$ |
| 5 | $w_2$ " " $w_3$ | $w_3$ " $w_6$ | $w_2$ " $w_6$ |
| 6 | $w_6$ " " $w_4$ | $w_4$ " $w_1$ | $w_6$ " $w_1$ |

From this table it can be seen that the function w acquires its original value after six consecutive circuits around a and c, and not before. If we think of the Riemann surface which consists of six sheets lying horizontally with the sheets numbered consecutively beginning at the top, and starting at a place in the upper sheet let z describe circuits round a and c until the path returns to its initial point, the path would change from one sheet to another as is shown in figure 10. In

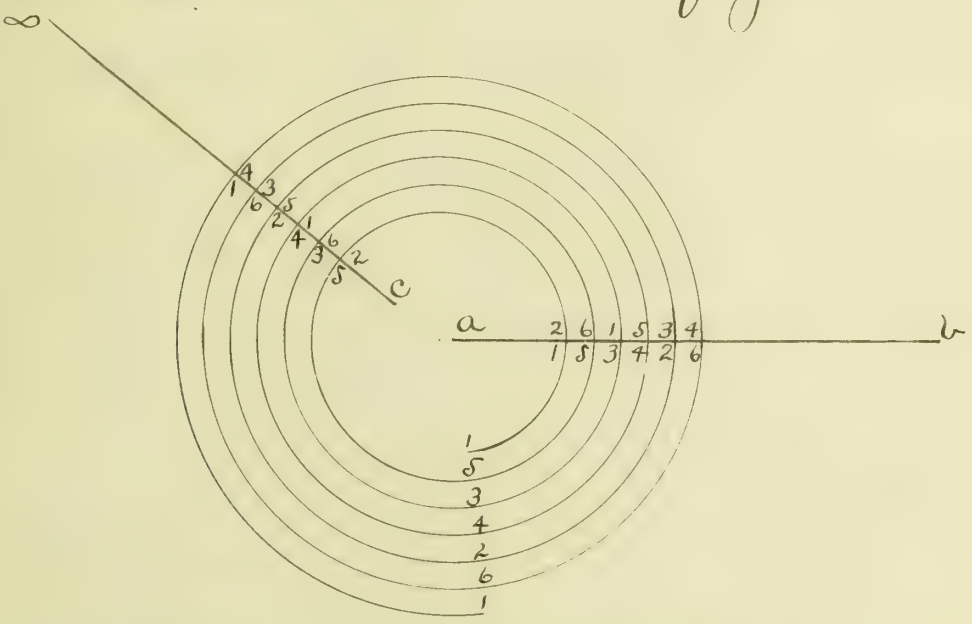


Fig 10

the figure, however, the points which lie below each other are placed side by side for the sake of clearness. The initial point 1 is supposed to coincide with the final point 1. The number of the sheet in which the path lies is designated by the figures along the branch cuts and at the points where the circuit begins and ends. It can be seen that a path which starts from an initial point $z_0$ in the first sheet, on meeting the branch cut from a to b, changes its position from the first into the second sheet, and at the cut from c to $\infty$ into the 5th sheet. Thus, in making the first circuit the path changes from the first into the 5th sheet. A second circuit changes it from the 5th into into the 3rd sheet, and corresponding changes are made for each circuit, until in the 6th circuit the path leaves the 6th sheet and returns to the

first. The curve is then closed and the initial value of the function is restored.

In a similar manner it is possible to determine the connection between the sheets for any Riemann surface as soon as the cyclic systems of the branch points are known.

15. Construction of the general Riemann surface. In the construction of the n sheeted Riemann surface for a given n valued function it is first necessary to find the values of z which locate the branch points in the z-plane. The value of z for these points can be found by eliminating w from the two equations,

$$f(w, z) = 0, \text{ and}$$

$$\frac{\partial f(w,z)}{\partial w} = 0,$$

as has been explained in section 13. Having done this, the next step is to draw the branch cuts. The most general way is to draw

lines, in the $z$-plane, from each of the branch cuts to infinity. However, as can be easily seen by the projection of the plane on the sphere, any other point, which is not a branch point, will answer the purpose equally well. Let $\zeta$ be such a point. Then branch cuts can be drawn from each of the branch points to $\zeta$. Let these branch cuts be $V_1, V_2, V_3$ etc. As seen by an observer at $\zeta$, let the right bank of these cuts be designated by $V_1^+, V_2^+, V_3^+$, etc., and the left banks by $V_1^-, V_2^-, V_3^-$, etc. When a path goes around $\zeta$ in a clockwise direction it meets these banks in the order $V_1^+, V_1^-, V_2^+, V_2^-$, etc. Now place $n$ sheets on the $z$-plane, each of which is identical with the one which has already been arranged, that is, the branch points, $\zeta$, and the cuts lie in the same position. It will next be necessary to find the cyclic systems by which

the the function values change when cir
cuits are made around a branch point.
The way in which these systems are deter
mined has been explained in the previous
section. When this is found out the banks
of the n-sheets, which lie together along
the cuts from the branch points to ∞,
may be connected in such a manner, that
a path can make the necessary change of
values during its circuit. Each positive
bank must be connected with the negative
bank of the sheet in which the following
consecutive values of the function lie. It
only remains to connect the n-sheets to-
gether at the branch points and the
surface is completed. However, when two
branch points have similar cyclic systems
it is possible to deform the two cuts,
which reach out to ∞ from the branch points,

into a single cut connecting the two points.

On the surface which has been formed according to this plan, any curve which restores the initial value of the function is a closed curve. For each value of the function there is one and only one corresponding place on the sheets, and for each place on the sheets there is one and only one value of the function. It can be seen that on the many sheeted Riemann surface the multi valued function is uniquely represented, and can be treated in the same way that a single-valued function is on the one sheet of the Argand plane.

# Chapter III

## Isothermic Surfaces

16. Separation of the function into its real and imaginary parts. We have already discussed a plan by which the multi-valued function of a complex variable can be given a geometric representation by means of the n-sheeted Riemann surface. We now take up the consideration of a second device, often used for the same purpose which, because of its most characteristic property, we will call the isothermic surface. In this case two separate surfaces are employed, one for the real and the other for the imaginary part of the function. Any function of a complex variable,

$$w = f(z),$$

in which

$$w = u + iv, \text{ and}$$

$$z = x + iy,$$

can be separated into two equations, one of which shows the equality between the real and the equality between the imaginary parts of the function. These equations can, by a simultaneous solution for $u$ and $v$, be arranged in the form,

(1) $$u = \phi(x, y)$$

(2) $$v = \psi(x, y).$$

Equation (1) is generally spoken of as the <u>real part</u>, and (2) as the <u>imaginary part</u> of the given function. Since these equations are functions of two real variables, they can be represented as surfaces by means of ordinary space coordinates in which $u$ or $v$ is one axis, and $x$ and $y$ the other two.

In the discussion which follows, w and v will always be considered as the vertical axis and x and y the axes in the horizontal plane, as in ordinary Cartesian coordinates.

17. Lines of equal potential and lines of streaming. If, for the real part of the function, we let w equal a constant, a plane is defined in space. This plane will be horizontal and parallel to the xy-plane, above which it lies at a distance equal to the constant w. On this plane a curve or set of curves will be determined when the parameters x and y take on different values. It follows, then, that as w takes on different constant values a series of parallel, horizontal planes will be formed, and in each plane will be a corresponding curve or set of curves in x and y. As the number of these planes grows larger and larger the difference between the consecutive values of w

grows smaller and smaller, until, finally in the limit the aggregate of the curves for constant values of $u$ will form a surface. The surface which is formed in this way is called the surface of reals, or surface of the real part of the function. For all finite regions in which the function is continuous the surface is continuous and for values which make the function discontinuous there will be found discontinuities in the resulting surface. The corresponding surface of imaginaries for the imaginary part of the function can be formed in a similar manner.

We see that each multi-valued function is represented by two distinct surfaces. Either of these surfaces is cut by the horizontal plane, determined by a particular constant value of $u$ or $v$, in a curve. These curves, because of important physical properties which they

possess, are called lines of equal potential.(x)
In any given isothermic surface the equal potential lines may be considered as the traces in the surface made by a series of parallel, horizontal curves which cut the surface. The curves which cut the system of equal potential lines orthogonally are called the lines of streaming. The projection of the surfaces on a horizontal plane, say the one for which the constant is equal to zero, carries over the two systems of curves of each surface so that the projected systems still cut each other orthogonally. In this way it is possible to map the lines of equal potential and lines of streaming of a particular surface into corresponding orthogonal systems which lie in a plane.

(x) These lines correspond to the "Niveaulinien" and "Fallinien" as used by the German mathematicians.

It will now be shown that if the two surfaces determined by a particular function be projected on the same plane, the lines of equal potential for the surface of reals will correspond to the lines of streaming for the surface of imaginaries, and the equal potential lines for the surface of imaginaries will correspond to the lines of streaming for the surface of reals. This condition implies that the two systems of equal potential lines are orthogonal to each other, and the same is implied for the lines of streaming. The proof can, therefore be based on the conditions for orthogonality, and on the definition which has been given for a regular function of a complex variable. The condition that two curves shall cut each other at right angles is, that at the point of intersection the derivative of one curve shall be equal to the negative reciprocal

of the derivative of the other curve. The derivative for one curve is $\frac{dy}{dx}$, and that for the other is $-\frac{1}{\frac{dy}{dx}}$. Hence, it follows that

$$\frac{dy}{dx} = \frac{1}{-\frac{dy}{dx}} = -\frac{dx}{dy}. \tag{1}$$

For the surface of reals, whose equation is

$$w = \phi(x, y)$$

$$\frac{dy}{dx} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial y}}, \tag{2}$$

and for the imaginary part of the function whose equation is

$$v = \psi(x, y)$$

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}. \tag{3}$$

The conditions for the existance of a function of a complex variable include the following differential equations (See section 1),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{4}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5}$$

Substituting these values in equation (3) gives.

$$\frac{dy}{dx} = \frac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial x}}. \tag{6}$$

Equation (6) is the negative reciprocal of the

value of $\frac{dy}{dx}$, as given in equation (2). Hence, equation (1) is satisfied and the two systems of equal potential lines, when projected on the same plane, are orthogonal to each other. Similar results may be obtained for the two systems of stream lines. Consequently, it may be said that the equal potential lines for the surface of reals correspond to the lines of streaming for the surface of imaginaries, and that the lines of streaming for the surface of reals correspond to the lines of equal potential for the surface of imaginaries, and conversely.

18. Application of Power Series to the Determination of tangential planes, cross points, and pinch-points. At any point $x$ which is not a singular point, it is possible to expand a multivalued function,

$$w = f(z),$$

in a power series having $(z-\alpha)$ appear with increasing, positive, consecutive integral exponents[x] The series is usually obtained by means of Taylor's development and is of the general form,

$$w - w_0 = f(z) - f(\alpha) = a(z-\alpha) + b(z-\alpha)^2 + c(z-\alpha)^3 + \cdots,$$

in which the coefficients $a, b, c$, etc. are respectively the first, second, third, etc derivatives of $w$ with respect to $z$. That such an expansion holds for a multi-valued function as well as for a single-valued function, can be seen at once by a glance at the Riemann surface. On such a surface the function can be treated as though it were single-valued, and consequently admits of expansion into a series the same as any single valued function. However, it

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[x] Burkhardt, Analytischen Functionen Einer Complex Veränderlichen, Section 37.

must be kept in mind that a particular series is for only one of the values of the function obtained for a given value of $z$. Each one of the several function values has its own particular series. For example, the two-valued function,

$$w = \sqrt{z},$$

has one expansion for $+\sqrt{z}$ and another for $-\sqrt{z}$. For a given value of $z$, say $\alpha$ the two series are,

$$w = +\sqrt{z} = \sqrt{\alpha}\left\{1 + \frac{1}{2}\frac{z-\alpha}{\alpha} - \frac{1}{8}\frac{(z-\alpha)^2}{\alpha^2} + \frac{1}{16}\frac{(z-\alpha)^3}{\alpha^3} - + \cdots\right\}$$

$$w = -\sqrt{z} = -\sqrt{\alpha}\left\{1 + \frac{1}{2}\frac{z-\alpha}{\alpha} - \frac{1}{8}\frac{(z-\alpha)^2}{\alpha^2} + \frac{1}{16}\frac{(z-\alpha)^3}{\alpha^3} - + \cdots\right\}$$

In general for an $n$-valued function there are $n$ different expansions which correspond to the $n$ different values of the function for a given value of the variable.

From the possibility of this development it follows that at a non-singular point $\alpha$, of a given function, there exists for both parts

of the surface definite tangential planes. These planes make with the horizontal plane an angle whose tangent is equal to the absolute value of a, that is, the coefficient of the first power of $(z-\alpha)$ which is the first derivative of the function. Moreover, when the two surfaces are formed with respect to the same set of space coordinates, the traces of these two tangential planes in the horizontal plane will cut each other at right angles. This can be seen by letting the horizontal plane pass through the point of tangency. Then the traces in the plane will coincide with the lines which are tangent to the two potential lines which pass through the point $\alpha$. From section 14, it follows that these two tangents cut each other at right angles. Consequently, the traces of the tangential planes in this particular horizontal

plane are also at right angles to each other. It is easy to see that the same principle holds for any other horizontal plane.

We shall next consider those points in the surface for which the first power of the series vanishes, or what is the same, a is equal to 0, while b, c, etc. are different from 0. In the first place, it can be seen that at such a point the tangential planes for both the real and imaginary surfaces are horizontal planes, since the tangent of the angle of inclination to a fixed horizontal plane, and consequently the angle itself, are equal to 0. At a point of this character the series takes the form,

$$w - w_0 = w - f(\alpha) = (z-\alpha)^2\{b + c(z-\alpha) + d(z-\alpha)^2 + \cdots\},$$

It is easy to see that $\alpha$ is a double root of the function. On both parts of the function's surface, at the point $\alpha$, two potential and

two stream lines intersect. Moreover, they will cut each other at equal angles. Such a point may be considered as the limiting case of two ordinary points of the surface as they gradually approach each other, and is called a <u>cross-point of multiplicity 2,</u> or <u>a cross point of the second order.</u> As a more general case we can take a point at which the first n terms of the series vanishes. Then the development has the form,

$$w - f(\alpha) = (z-\alpha)^n \{\xi + \eta(z-\alpha) + \zeta(z-\alpha)^2 + \cdots\}$$

It follows that n potential lines and n stream lines cut each other at equal angles at the point $\alpha$. The point is a cross point of the $n^{th}$ order. Should all of the different powers of $(z-\alpha)$ vanish at a point on the surface, we have a cross point of infinitely high order, or an <u>essentially singular point.</u>

The branch points which are found in

the Riemann surface for a given function, generally appear in the isothermic surfaces for the same function as <u>finite points</u>. At such points the first derivative of the function with respect to $z$ is infinite, and the tangential planes make an angle of $\frac{\pi}{2}$ with the horizontal plane. The position of the finite points in the $xy$ plane can be determined as the branch points were for the Riemann surface by eliminating $w$ from the two equations,

$$f(w, z) = 0, \text{ and}$$
$$\frac{\partial f(w, z)}{\partial w} = 0,$$

(At such points, the powers of $(z - \alpha)$ which are obtained by expanding the function into a power series have increasing, but fractional exponents. If the branch point is a pole, the exponents are both positive and negative, the number of negative exponents

being equal to the order of the pole.[x] The series for the case where there is no pole has the general form,

$$w - f(\alpha) = a(z-\alpha)^{\frac{1}{q}} + b(z-\alpha)^{\frac{2}{q}} + c(z-\alpha)^{\frac{3}{q}} + \cdots,$$

in which $q$ corresponds to the number of values the function takes for a given value of $z$, and $\alpha$ is the branch branch point.

After this general discussion of the properties of isothermic surfaces, the representation of a few typical functions will be taken up. The surfaces which will be discussed are those of Series XIV, Bill's models, which are possessed by the Mathematical Department of the University of Illinois.

19. The function $w^2 = z^2 - 1$. One of the simplest forms of the isothermic surfaces is the one which is determined by the algebraic

[x] Burkhardt, Analytischen Functionen einer Complex Veränderlichen, Section 61.

function,

(1) $$w = z^2 - 1,$$

It is represented by surfaces Ia and Ib of the Bill Models.

To separate the function into its real and imaginary parts, we put

(2) $$w = u + iv, \text{ and}$$

$$z = x + iy.$$

Then, on making these substitutions

(3) $$u^2 - v^2 + 2iuv = x^2 - y^2 - 1 + 2ixy$$

Separating into real and imaginary parts, we obtain

(4) $$u^2 - v^2 = x^2 - y^2 - 1$$

(5) $$uv = xy.$$

These equations, however, are not in a form which answers our purpose, because they each contain $u$ and $v$. From (5),

$$v = \frac{xy}{u}.$$

Substituting this value of $v$ in (4), we obtain

(6) $$w^4 - (x^2 - y^2 - 1)w^2 - x^2y^2 = 0.$$

By a similar substitution for $w$, there is obtained

(7) $$v^4 + (x^2 - y^2 - 1)v^2 - x^2y^2 = 0.$$

Equation (6) is the equation of the surface of reals, or the $w$ surface, and (7) gives the surface of imaginaries or the $v$ surface.

Let us first investigate these surfaces for cross and branch points. Placing the given function in the form,

$$w \sqrt{z^2 - 1},$$

and expanding by Taylor's development into a series having increasing, positive, integral exponents of $(z - z_0)$, we obtain

$$w - w_0 = \frac{z_0}{(z_0^2 - 1)^{\frac{1}{2}}}(z - z_0) - \frac{1}{2(z_0^2 - 1)^{\frac{3}{2}}}(z - z_0)^2 - \frac{z_0}{3!(z_0^2 - 1)^{\frac{5}{2}}}(z - z_0)^3 + \cdots.$$

When $z_0$ becomes zero, the coefficient of the first power of $(z - z_0)$ vanishes, while the coefficient of the second power is different from zero and finite. From this it follows that for both parts of the surface the point zero

is a cross point of the first order.

By inspection it can be seen that the two values of $w$ are equal when $z$ is equal to $\pm 1$. Hence, $\pm 1$ in the isothermic surfaces corresponds to the branch points of the Riemann surface. The points on the $x$ axis, at a unit's distance on both sides of the origin, will appear as finite points. When the function is expanded in fractional powers of $z \pm 1$, it will have the form,

$$w = \sqrt{z^2 - 1} = a(z \mp 1)^{\frac{1}{2}} + b(z \mp 1)^{\frac{3}{2}} + \cdots,$$

which is another proof that $\pm 1$ is a finite point, since the expansion is true for

$$z = \pm 1.$$

In order to get the general form of the surface let us find the traces which are made on the coordinate planes. We will first take up the real part (6). Solving equation (6) with the equation for the $xy$-plane,

for which,

$$x = 0,$$

there is obtained

$$w^4 + (y^2+1)w^2 = 0.$$

This can be factored into two parts,

$$w^2 = 0, \text{ and}$$

$$w^2 + y^2 = -1.$$

We have, then, a pair of straight lines coincident with the y-axis and an imaginary circle of radius $\sqrt{-1}$. Since the representation depends entirely on real variables the circle does not appear in the surface.

Cutting the surface with the wx-plane, for which

$$y = 0,$$

there is obtained

$$w^4 - (x^2-1)w^2 = 0,$$

which can be factored into

$$w^2 = 0, \text{ and}$$

$$w^2 - x^2 = 1.$$

The first equation gives two lines which are coincident with the x- axis. The second one is the equation of a rectangular hyperbola with its vertices at the points on the x- axis where x equals $\pm 1$, and asymtotic to the lines,

$$w = \pm x.$$

The trace in the xy- plane is given by the equation,

$$x^2 y^2 = 0$$

which represents two pairs of straight lines which are coincident with the x- and the y- axes.

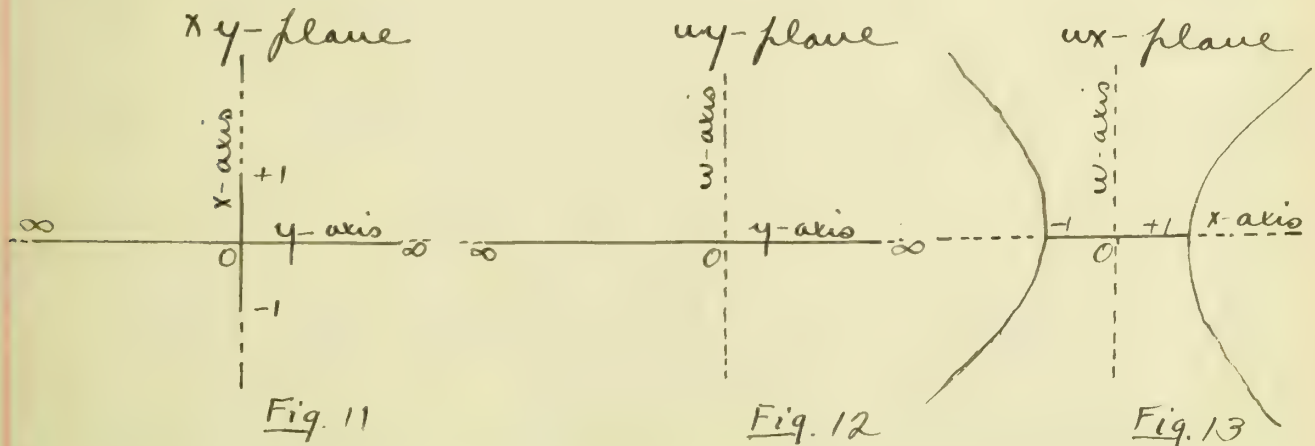


Fig. 11 Fig. 12 Fig. 13

Figures 11-13 show the traces of the surface on the three coordinate planes.

The form of the completed surface is obtained by giving to w a series of constant values, and plotting the curves defined by x and y in the various planes. The limit as these Equal potential lines become closer and closer defines the completed surface. The table of values given on the following page locates the points on the surface which lie near the origin. If the two orthogonal systems of curves should be plotted on the same plane, or if they were all projected on the xy-plane, the essential characteristics of the lines can be shown. The lines of streaming are determined by constructing a system of curves orthogonal to the potential curves. Figure 14 shows such a mapping of the two orthogonal systems, as projected on one of the horizontal planes.

# Curves where w is given constant values.

| w | y | x |
|---|---|---|
| 0 | 0 | 0 |
| $\pm \frac{1}{3}$ | 0 | $\pm \sqrt{\frac{10}{9}}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{19}{90}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{46}{333}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{91}{738}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{154}{1305}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{235}{2061}}$ |
| $\pm \frac{2}{3}$ | 0 | $\pm \sqrt{\frac{13}{9}}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{88}{117}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{248}{360}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{428}{765}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{628}{1332}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{952}{2061}}$ |
| $\pm 1$ | 0 | $\pm \sqrt{2}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{3}{2}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{6}{5}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{11}{10}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{18}{17}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{27}{26}}$ |
| | $\pm 6$ | $\pm \sqrt{\frac{38}{37}}$ |
| $\pm \frac{4}{3}$ | 0 | $\pm \frac{5}{3}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{544}{225}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{976}{468}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{1296}{813}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{2304}{1440}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{4000}{2169}}$ |
| $\pm \frac{5}{3}$ | 0 | $\pm \sqrt{\frac{34}{9}}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{1075}{306}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{1650}{549}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{2875}{1054}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{4450}{1521}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{6475}{2250}}$ |
| $\pm 2$ | 0 | $\pm \sqrt{5}$ |
| | $\pm 1$ | $\pm \sqrt{\frac{24}{5}}$ |
| | $\pm 2$ | $\pm \sqrt{\frac{9}{2}}$ |
| | $\pm 3$ | $\pm \sqrt{\frac{56}{13}}$ |
| | $\pm 4$ | $\pm \sqrt{\frac{86}{20}}$ |
| | $\pm 5$ | $\pm \sqrt{\frac{120}{29}}$ |
| | $\pm 6$ | $\pm \sqrt{\frac{164}{40}}$ |

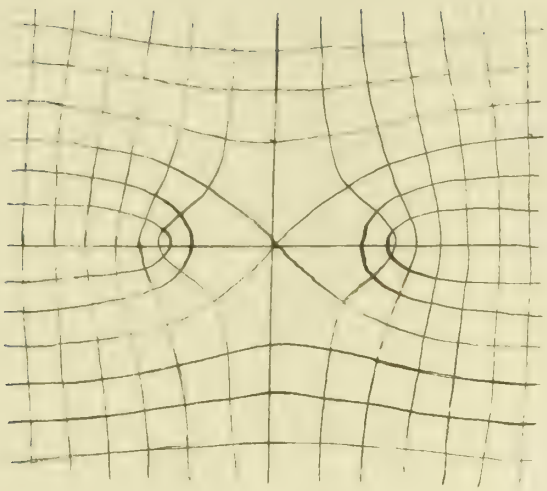

Fig. 14

In the figure the lines which run in a general vertical direction are the equal potential lines for the surface of reals and the lines of streaming for the surface of imaginaries, while those in a general horizontal direction are the lines of streaming for the surface of reals and the equal potential lines for the surface of imaginaries. It can be seen that a cross joint appears at the origin, and that $\pm 1$ are pinch points, as has already been shown analytically. It can also be noted that each of the lines

of equal potential has an asymtote. The equation of the asymtote being obtained by giving to $x$ the same absolute value as that which belongs to $u$ for the plane in which the curve lies. For example, take the plane whose absolute value is equal to 1. The equation of the $xy$ curve in this plane is

$$x^2 + x^2y^2 - y^2 - 2 = 0, \tag{1}$$

which is asymtotic to the lines,

$$x = \pm 1. \tag{2}$$

This can be seen by an inspection of the values which $x$ takes as $y$ takes continually increasing values, or by changing the infinite values to zero by geometric inversion and showing that the resulting curves are tangent at 0. Substituting in (1) the reciprocal value of $y$, and clearing of fractions, we obtain

$$x^2y^2 + x^2 - 1 - 2y^2 = 0. \tag{3}$$

This equation is satisfied for the values

$$x = 1, \text{ and}$$

$$y = 0,$$

which before the transformation by inversion were 1, and infinity respectively. The equation of the asymtote is also satisfied for the same values, which shows that the two curves touch each other at the point (1, 0). On changing back to the original form it can be seen that the line,

$$x = \pm 1,$$

is an asymtote. From this it follows that the surface of reals is asymtotic to the planes whose equations are

$$w = \pm x.$$

We shall now take up the consideration of the surface of imaginaries whose Equation has been shown to be

$$v^4 + (x^2 y^2 - 1) v^2 - x^2 y^2 = 0.$$

Proceeding as for the real surface we first find the traces of the surface made on the coordinate planes. Solving the equation with the equation of the $vx$- plane, there is obtained

$$v^4 + (x^2 - 1)v^2 = 0,$$

which can be separated into

$$v^2 = 0, \text{ and}$$

$$v^2 + x^2 = 1.$$

The first equation represents a pair of straight lines which are coincident with the $x$- axis. The second equation is a circle about the origin with a unit radius.

Cutting the surface with the $vy$- plane for which

$$x = 0,$$

there is obtained the equation,

$$v^4 + (-y^2 - 1)v^2 = 0,$$

which can be factored into

$$v^2 = 0, \text{ and}$$

$$v^2 - y^2 = 1$$

The first equation is represented by a pair of straight lines coincident with the y-axis. However, a comparison with the traces made on the other coordinate planes shows that these lines do not enter into the surface. The second equation is a rectangular hyperbola whose asymtotes are the lines,

$$v = \pm y.$$

The traces of the xy-plane are given by the equation

$$x^2 y^2 = 0.$$

This Equation is represented by two pair of straight lines coincident with the x and y axes.

Figures 15 17 show the traces which are made on the three coordinate planes. The completed surface is obtained, as for the surface of rials by giving to $v$ a series of constant

# Curves when $v$ is given constant values.

| $v$ | $x$ | $y$ |
|---|---|---|
| 0 | 0 | 0 |
| $\pm \frac{1}{3}$ | 0 | imag. |
| | $\pm 1$ | $\pm\sqrt{\frac{1}{90}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{28}{933}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{73}{738}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{136}{1305}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{217}{2034}}$ |
| $\pm \frac{2}{3}$ | 0 | imag. |
| | $\pm 1$ | $\pm\sqrt{\frac{16}{117}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{124}{960}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{304}{765}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{556}{1332}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{880}{2061}}$ |
| $\pm 1$ | 0 | 0 |
| | $\pm 1$ | $\pm\sqrt{\frac{1}{2}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{4}{5}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{9}{10}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{16}{17}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{25}{26}}$ |
| $\pm \frac{4}{3}$ | 0 | $\pm\sqrt{\frac{167}{9}}$ |
| | $\pm 1$ | $\pm\sqrt{\frac{16}{15}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{688}{468}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{1408}{873}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{2416}{1440}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{3712}{2169}}$ |
| $\pm \frac{5}{3}$ | 0 | $\pm\frac{20}{15}$ |
| | $\pm 1$ | $\pm\sqrt{\frac{625}{306}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{1300}{549}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{2425}{1054}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{3600}{1521}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{6025}{2250}}$ |
| $\pm 2$ | 0 | $\pm\sqrt{3}$ |
| | $\pm 1$ | $\pm\sqrt{\frac{16}{5}}$ |
| | $\pm 2$ | $\pm\sqrt{\frac{7}{8}}$ |
| | $\pm 3$ | $\pm\sqrt{\frac{48}{13}}$ |
| | $\pm 4$ | $\pm\sqrt{\frac{26}{20}}$ |
| | $\pm 5$ | $\pm\sqrt{\frac{112}{29}}$ |

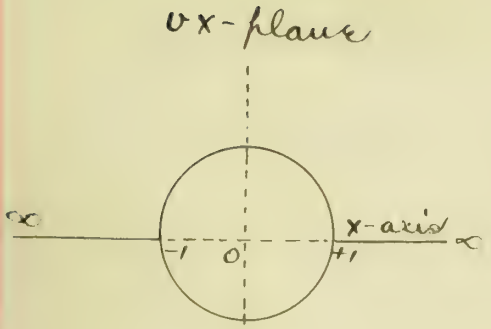


Fig. 15

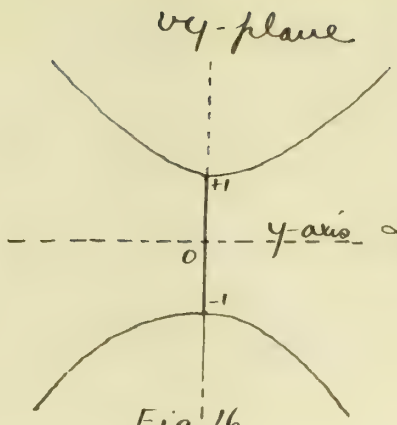


Fig. 16

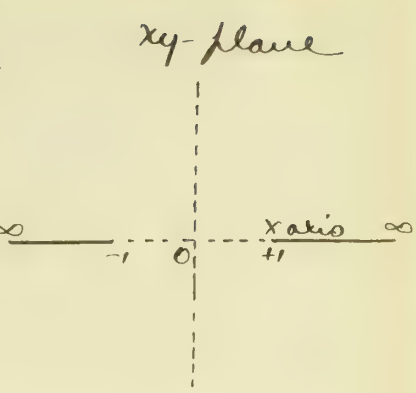


Fig. 17

values and plotting the xy curves in the resulting xy-planes. The limit of these equal potential lines as the horizontal planes become closer and closer will define the completed surface. The table of values given on the preceding page shows the points of the surface near the origin, as projected from horizontal planes one third unit apart. Figure 14 shows the mapping of the two orthogonal systems on a horizontal plane. The table of values for the two surfaces shows that the cross points for the surface of imaginaries are found in the planes for which v equals $\pm 1$, while those for the surface of reals is in the plane for which v is equal to 0.

20 Function $w^2 = z^4 - 1$. We shall now take up the discussion of a second algebraic, two-valued function, which closely resembles the one just considered in its most important features. This function is

$$(1) \qquad w^2 = z^4 - 1.$$

It is represented by models numbered II a and II b of Bill's series XIV. This surface differs essentially from the previous function in that it has a cross point of the fourth order, and four corresponding quick points.

We first separate the function into its real and imaginary parts. Let

$$w = u + iv, \text{ and}$$

$$z = x + iy.$$

Substituting in (1), we have

$$(2) \qquad u^2 - v^2 + 2iuv = x^4 + 4ix^3y - 6x^2y^2 - 4ixy + y^4$$

Equating real and imaginary parts, we obtain

$$(3) \qquad u^2 - v^2 = x^4 + y^4 - 6x^2y^2, \text{ and}$$

(4) $$uv = 2x^3y - 2xy^3.$$

From (4), it follows that

(5) $$u = \frac{2x^3y - 2xy^3}{v}, \text{ and}$$

(6) $$v = \frac{2x^3y - 2xy^3}{u}.$$

Substituting (5) and (6) in (3), we have

(7) $$u^4 - (4x^4 - 6x^2y^2 + y^4 - 1)u^2 - 4x^2y^2(x^2 - y^2)^2 = 0$$

(8) $$v^4 + (4x^4 - 6x^2y^2 + y^4 - 1)v^2 - 4x^2y^2(x^2 - y^2)^2 = 0$$

Equation (7) determines the surface of reals and (8) the surface of imaginaries.

In order to examine for cross points, we shall express the equation as a function of $z$,

$$w = \sqrt{z^4 - 1},$$

and expand it into a series, by Taylor's development, at the point $\alpha$. In this way we obtain,

$$w - w_0 = \sqrt{z^4 - 1} - \sqrt{\alpha^4 - 1} = \frac{2\alpha^3}{(\alpha^4 - 1)^{\frac{1}{2}}}(z - \alpha) + \frac{2\alpha^2(\alpha^4 - 3)}{(\alpha^4 - 1)^{\frac{3}{2}}}(z - \alpha)^2 +$$

$$\frac{12\alpha(\alpha^4 + 1)}{(\alpha^4 - 1)^{\frac{5}{2}}}(z - \alpha)^3 - \frac{12(5\alpha^8 + 14\alpha^4 + 1)}{(\alpha^4 - 1)^{\frac{7}{2}}}(z - \alpha)^4 + \cdots,$$

If $\alpha$ is equal to 0 the first powers of the series vanish and the expansion has the form,

$$w - w_0 = -1 z^4 + \cdots$$

This shows that the point zero is a cross point of the fourth order, since the exponent of the first power which appears in the development is the fourth.

To locate the finite points we need only to find the points at which the two values of the function are equal. By factoring (1), we obtain

$$w = \sqrt{z-1}\,\sqrt{z+1}\,\sqrt{z^2+1},$$

which shows that the values of $w$ are equal for

$$z = \pm 1, \text{ and}$$
$$z = \pm i.$$

Consequently the finite points are located on each half axis of the $xy$ plane at a unit's distance from the origin.

Having located the singular points, we shall now develop the surface for the real part of the function, Equation (1).

The trace on the $ux$-plane is determined by the equation,

$$w^4-(x^4-1)w^2=0,$$

which can be factored into

$$w^2=0, \text{ and}$$

$$w^2-x^2=1.$$

The first part of the equation is represented by two lines coincident with the x-axis. The second part is an hyperbola with its vertices at the branch points on the x-axis. Figure 18 shows the traces on the wx-plane.

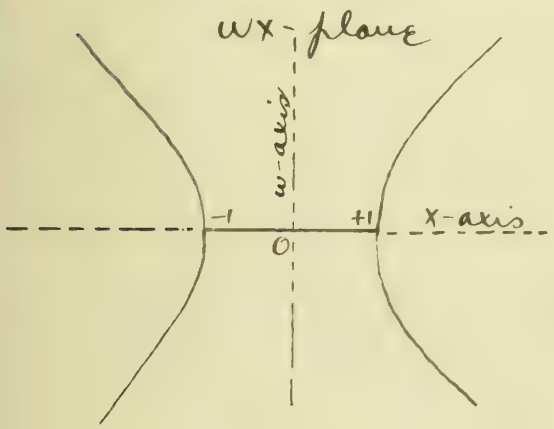


Fig 18

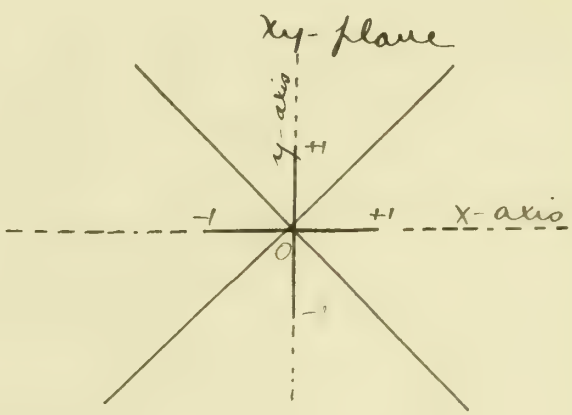


Fig 19

Since the equation is symmetrical with respect to x and y, similar traces will be made on the wy-plane. The traces on the xy-plane are given by the equation,

$$x^2y^2(x^2-y^2)=0,$$

which separates into

$$x^2y^2=0, \text{ and}$$

$$x^2-y^2=0.$$

The first equation gives a pair of straight lines coincident with the axes of the plane, and the second two straight lines which bisect the angles between the xy coordinates. Figure 19 shows the position of these lines.

The traces which are made on the planes, which bisect the angle between the wx and wy coordinate planes, need special attention. They are defined by the equations,

$$x=\pm y, \text{ or } x^2=y^2.$$

Solving simultaneous with the equation for the surface we obtain,

$$w^4+4x^4w^2=0, \text{ and}$$

$$w^4+4y^4w^2=0,$$

which can be factored into,

$$w^4 = 0,$$

$$w^2 + 4x^4 = 0,$$

$$w^2 + 4y^4 = 0.$$

The first equation gives two double lines which lie in the $xy$-plane and bisect the angle between the $x$ and $y$ axes. The other two equations give imaginary curves. It follows, then, that the surface is divided into four parts by the lines in the $xy$ plane, whose equations are

$$w = 0, \text{ and } x = \pm y.$$

For the imaginary part of the surface we have equation (8). Its trace on the $vx$-plane is given by the equation,

$$v^4 + (x^4 - 1)v^2 = 0,$$

which may be factored into,

$$v^2 = 0, \text{ and}$$

$$v^2 + x^4 = 1.$$

This gives a pair of straight lines coincident with the $x$-axis, and a higher plane curve

as shown in figure 20.

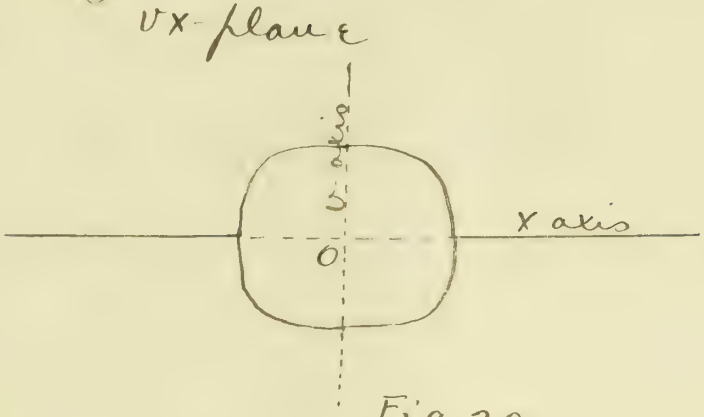


Fig. 20

Cutting the surface by the vy-plane, we obtain

$$v^4 + (y^4 - 1)v^2 = 0,$$

which gives traces similar to those for the vx-plane.

The traces on the xy-plane are given by the equation,

$$4x^2y^2(x^2 - y^2)^2 = 0,$$

which separates into

$$x^2y^2 = 0, \text{ and}$$

$$(x^2 - y^2)^2 = 0.$$

These equations are represented by double lines coincident with the x and y axes, and another

pair of lines which do not enter into the continuous surface.

In order to plot the surface in detail, as for the function in section 16, we first give consecutive values to $u$ and $v$, and then find the resulting potential lines. By projecting the two systems of potential lines on the same

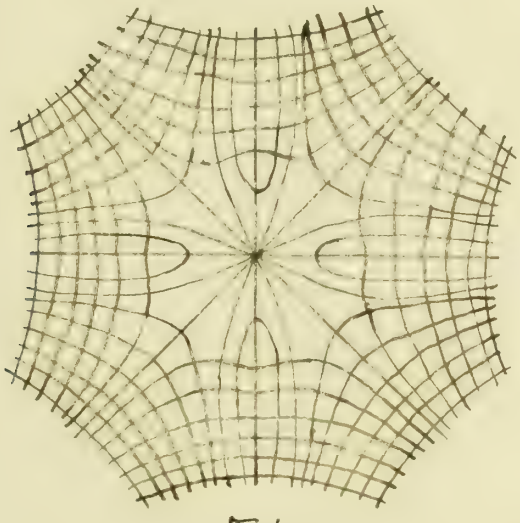

Fig 21

horizontal plane, we obtain the projection of the orthogonal curves for either part of the surface. The map shown in figure 21 gives the curves near the origin when the values of $u$ and $v$ increase by multiples of one third of a unit.

We shall now consider the important relation which the given function bears to the function,

(1) $$w = \pm z^2, \text{ or } w^2 = z^4.$$

In order to examine the function at the point infinity, we make use of the reciprocal transformation and substitute $\frac{1}{z}$ for $z$. We then obtain for the two functions,

(2) $$z^4 w^2 = 1 - z^4, \text{ and}$$

(3) $$z^4 w^2 = 1.$$

This shows that the two reciprocal functions have equal values at the origin. Changing back to the original functions, shows that the two equations have the same values at infinity. If we make the usual substitutions, and separate the function into its real and imaginary parts, we obtain from (1),

(5) $$u = \pm (x^2 - y^2),$$

(6) $$v = \pm 2xy,$$

From this it follows that the surface of reals

touches the hyperbolic paraboloids, as determined by Equation (5), at infinity, and the surface of imaginaries, the paraboloids whose equations are (6), at infinity.

The two functions which have just been discussed are but special cases of the more general function,

$$w^2 = z^n - 1,$$

in which $n$ may be any positive integral number. The general has a cross point of the $n^{th}$ order at zero, and such points at unit's distance from the origin, on straight lines, which make equal angles with each other, in the $xy$-plane.

21. Function $w^4 = 1 - z^2$. We now take up the discussion of a four-valued function,

$$w^4 = 1 - z^2.$$

It will be seen at once that this equation is the same as the one just considered, if the

variables $z$ and $w$ be interchanged. The function is also periodic in that its real and imaginary parts are identical, as the following discussion will show. If the following transformations are made,

$$w' = \pm iw, \text{ and}$$
$$z' = \pm z,$$

the function goes over into itself. Thus, if we give to $z$ a certain value, say $\pm z_0$, we can put for $w$ the value $\pm w_0$ and $\pm iw_0$. We can give to $z$ the values,

$$z = \pm(x + iy),$$

and the corresponding values of $w$ will be

$$w = (u+iv),\ -(u+iv),\ (-v+iu),\ \text{and}\ -(-v+iu).$$

Since these values contain all possible combinations of $x$ and $y$, and $u$ and $v$, and they all give the function the same value, it follows that the surfaces of reals and the surfaces of imaginaries are identical. This surface is

number III of the Brill models.

We will now proceed to separate the function into its real and imaginary parts, by substituting,

$$w = u + iv, \text{ and}$$

$$z = x + iy.$$

In this way, we obtain,

$$u^4 + 4iu^3v - 6u^2v^2 - 4iuv^3 + v^4 = 1 - x^2 + y^2 - ixy,$$

which can be separated into,

$$u^4 - 6u^2v^2 + v^4 = 1 - x^2 + y^2; \text{ and}$$

$$2u^3v - 2uv^3 = -xy.$$

Solving the second equation for $u$ and $v$, and substituting in the first one we obtain the equation for $w$,

$$256w^{16} - 128w^{12}(-x^2+y^2+1) - 16w^8\}(x^4+y^4-2x^2 + 2y^2+1) + 20x^2y^2\} - 8w^4\{2(x^4+y^4-2x^2+2y^2+1) + 3x^2y^2\}\{-x^2+y^2+1\} + x^4y^4 = 0.$$

If we substitute $v$ for $w$ in this equation, the equation for the surface of imaginaries is obtained.

If we expand the function by Taylor's series, we obtain

$$w - w_0 = \sqrt[4]{1-z^2} - \sqrt[4]{1-\alpha^2} = \frac{\alpha}{2(1-\alpha^2)^{3/4}}(z-\alpha) - \frac{1+2\alpha^2}{2(1-\alpha^2)^{\frac{7}{4}}}(z-\alpha)^2 -$$

The first term of this series vanishes when $\alpha$ becomes equal to zero. Hence, it follows that zero is a cross point of the second order. The function also has equal values for the points,

$$z = \pm 1 .$$

Therefore, the function has branch points for these values, and the four leaves of the surface are joined together at these points.

If the surface be cut by the plane

$$y = 0,$$

we obtain the curves which are defined by the equation

$$256\,w^{16} - 128\,w^{12}(1-x^2) - 16\,w^8\{7(x^4-2x^2+1)\}$$
$$-8\,w^4\{2(x^4-2x^2+1)\}\cdot\{-x^2+1\} = 0,$$

which can be written as

$$16\,w^4 = 0, \text{ and}$$

$$16w^{12} - 8w^8(1-x^2) - 7w^4(1-x)^2 - (1-x^2)^3 = 0.$$

The first equation is represented by a pair of coincident straight lines along the x axis. The second equation can be plotted by giving values to $(1-x^2)$ and solving for $w^4$. By this means the curve is found to be as shown in figure 22.

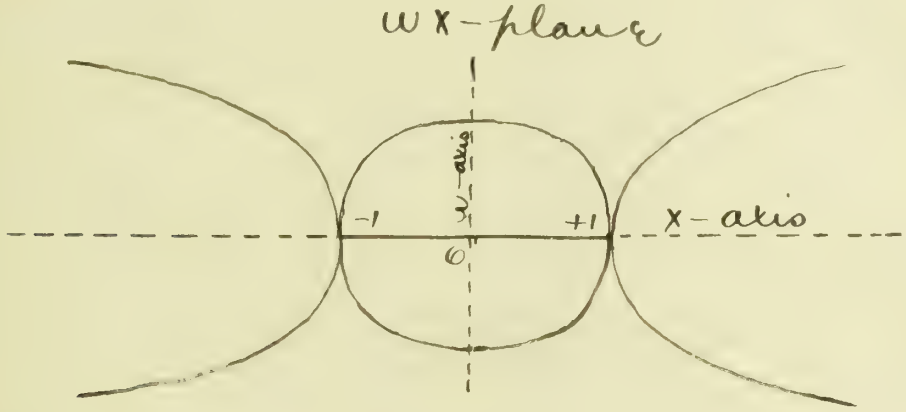


Fig 22.

For the traces on the plane,

$$x = 0,$$

we obtain

$$256w^{16} - 128w^{12}(y^2+1) - 16w^8\{7(y^4+2y^2+1)\}$$
$$- 8w^4\{2(y^4+2y^2+1)\}\cdot\{y^2+1\} = 0,$$

which separates into

$16w^4 = 0$, and

$16w^{16} - 8w^8(y^2+1) - 7w^4(y^2+1)^2 - (y^2+1)^3 = 0.$

The first of these equations represents a straight line coincident with the y axis, and the second can be plotted as a higher plane curve in the usual way. Figure 23 shows the resulting curves.

The trace on the xy plane is given by the equation

$$x^4 y^4 = 0,$$

which defines lines coincident with the x and y axes, as shown in figure 24.

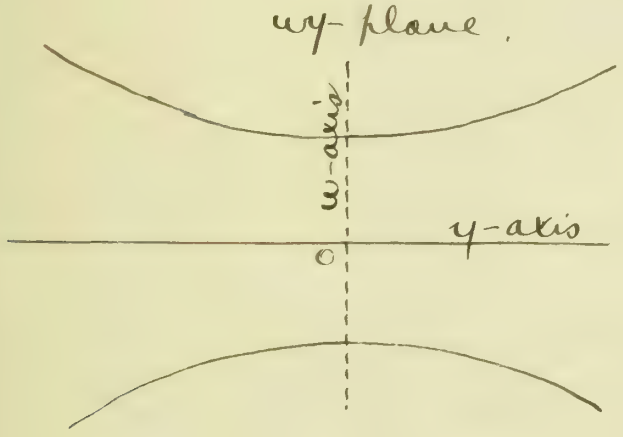


Fig. 23

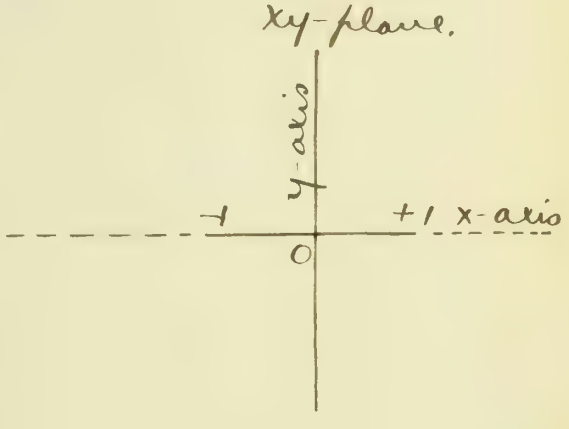


Fig. 24

By giving to w the constant values, $0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}$, etc., we are able to construct the

the equal potential lines which define the surface. As has already been shown the surface for v will be identical with the w surface. From this it follows that the equal potential lines, for either part of the surface, must cut themselves orthogonally. Figure 25 shows the projections of the systems, near the origin for the w or the v surface. It can be seen how

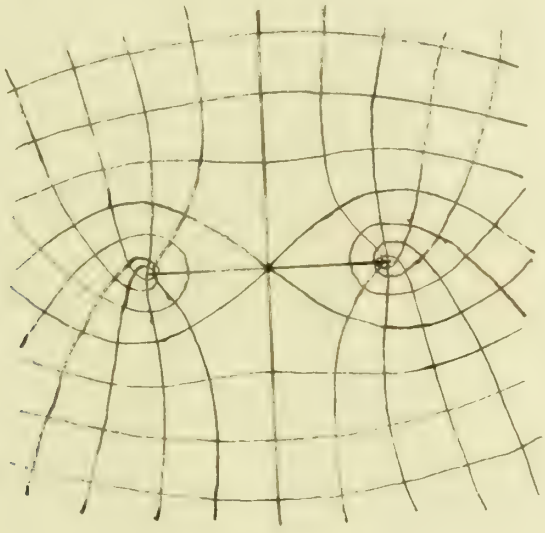

Fig. 25.

the lines of the same system cut themselves orthogonally.

22. Functions $u = \frac{1}{2}$ and $w = \frac{1}{2\varepsilon} \log \frac{z+\varepsilon}{z-\varepsilon}$.

We shall now leave the algebraic functions

and take up the discussion of some typical logarithmic surfaces. First, we shall consider an equation of the form,

$$(1) \qquad w = \frac{1}{2\varepsilon} \log \frac{z+\varepsilon}{z-\varepsilon}.$$

If this equation is given a more special form in which $\varepsilon$ is equal to zero, we have

$$(2) \qquad w = \frac{1}{z}.$$

However, if we substitute zero for $\varepsilon$, we obtain

$$w = \frac{1}{0} \log 1 = \frac{0}{0},$$

which is indeterminate. It can, however, be evaluated by putting the equation in the form,

$$(3) \qquad e^{2\varepsilon w} = \frac{z+\varepsilon}{z-\varepsilon},$$

and differentiating with respect to $\varepsilon$. We then have the derived equation,

$$(4) \qquad 2w e^{2\varepsilon w} = \frac{2z}{(z-\varepsilon)^2},$$

which can be evaluated for $\varepsilon$ equal to zero, and equation (2) is obtained.

Separating $w$ and $z$ into their real and

imaginary parts and substituting in (2), we obtain

(5) $$wx - ivx + iwy + vy = 1,$$

which can be separated into

(6) $$wx + vy = 1, \text{ and}$$

(7) $$xv = wy.$$

From (7), we have

(8) $$v = \frac{wy}{x}, \text{ and}$$

(9) $$w = \frac{vx}{y}.$$

Substituting (8) and (9) in (6), we obtain

(10) $$w(x^2+y^2) - x = 0, \text{ and}$$

(11) $$v(x^2+y^2) + y = 0.$$

The surface has no finite cross points, as no terms of the series,

$$w - w_0 = \frac{1}{2} - \frac{1}{\alpha} = -\frac{1}{\alpha^2}(2-\alpha) + \frac{2}{\alpha^3}(2-\alpha)^2 + \cdots,$$

vanishes for finite values of $\alpha$. However, at infinity we have a cross point of infinitely high order. It can also be seen that there are no branch points in the finite regions.

Proceeding with the real part of the function,

we find by solving (1) simultaneously with

$$y = 0,$$

the equation,

$$wx^2 - x = 0,$$

which gives the trace on the wx-plane. This can be separated into

$$x = 0, \text{ and}$$

$$wx = 1.$$

The first equation gives a straight line coincident with the w-axis, and the second a rectangular hyperbola, which is asymtotic to the w- and x- axes.

The trace on the wxy-plane is given by the equation,

$$wy^2 = 0,$$

which separates into

$$w = 0, \text{ and}$$

$$y^2 = 0.$$

We have, therefore, a line along the y-axis,

and a double line coincident with the w axis.

For the trace on the xy-plane, we have

$$x = 0,$$

which is the y axis. Figures 26-28 show

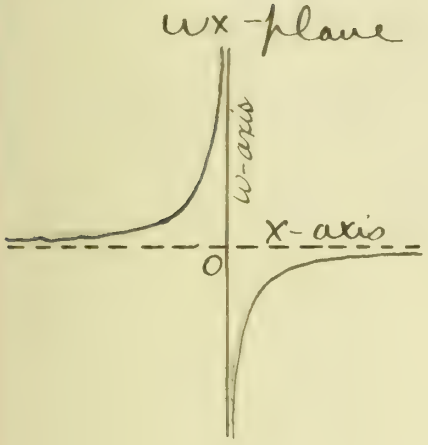


Fig 26

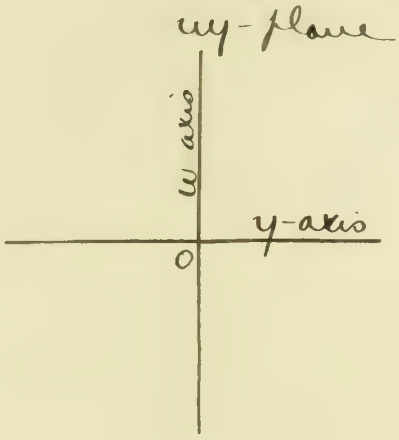


Fig 27

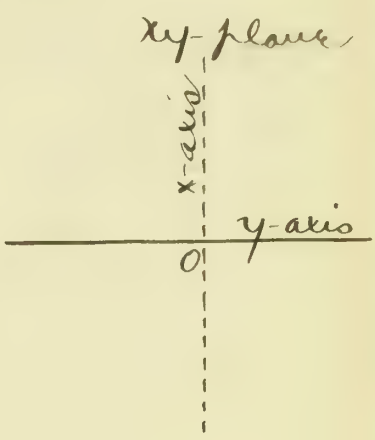


Fig 28

the traces on the three coordinate planes.

To find the equal potential lines which correspond to the different values of w, we place the equation in the form,

$$\left(x - \frac{1}{2w}\right)^2 + y^2 = \frac{1}{4w^2},$$

It is easy to see that for a constant value of w, there is obtained a circle with its center on the x-axis at a distance of $\frac{1}{2w}$ from the origin, and tangent to the wy-plane

along the w axis. It follows, then, that the projected system of equal potential lines, on the horizontal plane, will be a system of circles tangent to the y axis at the origin and with a radius equal to $\frac{1}{2}w$. The stream lines will be a similar system of circles tangent to the x axis at the origin. Figure 29

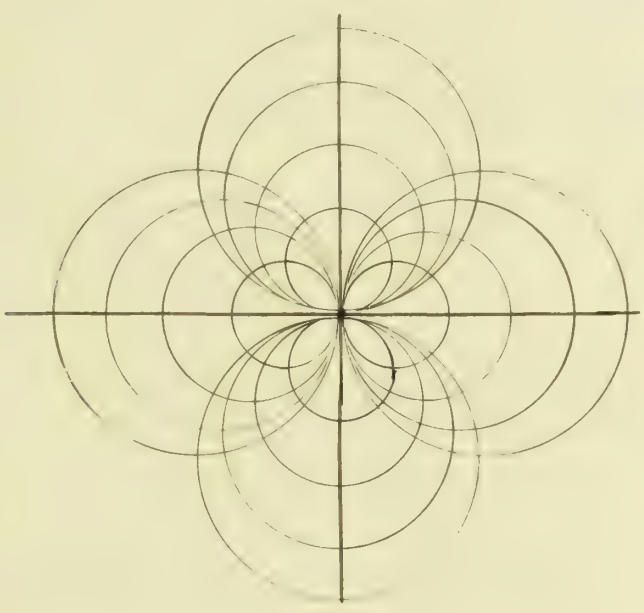

Fig 29

shows the mapping of the two orthogonal systems. The completed surface may be said to be generated by a circle of varying diameter, which moves along in a horizontal position,

keeping the extremities of its diameter on the w-axis and on the hyperbola defined by the equations,

$$y = c, \text{ and}$$
$$ux = 1.$$

If we turn the surface of revolution which has just been developed, through an angle of $\frac{\pi}{2}$ in the xy-plane we obtain the surface of imaginaries. This can be shown analytically by placing the equation in the polar form, by the substitutions,

$$w = r(\cos\phi + i\sin\phi), \text{ and}$$
$$z = \rho(\cos\theta + i\sin\theta).$$

These substitutions change the original equation into the form,

$$r(\cos\phi + i\sin\phi) = \frac{1}{\rho(\cos\theta + i\sin\theta)},$$
$$= \frac{1}{\rho}(\cos\theta - i\sin\theta).$$

On turning the plane through an angle of $\pi/2$, we change $\theta$ into $(\theta + \frac{\pi}{2})$. This sub-

stitution gives

$$r(\cos\phi + i\sin\phi) = \frac{1}{\rho}\{\cos(\theta+\tfrac{\pi}{2}) - i\sin(\theta+\tfrac{\pi}{2})\},$$
$$= \frac{1}{\rho}\{-\sin\theta - i\cos\theta\},$$
$$= \frac{i}{\rho}\{\cos\theta - i\sin\theta\},$$
$$= \frac{i}{\rho(\cos\theta + i\sin\theta)}.$$

Changing back into the original form, we obtain

$$w = \frac{i}{z}, \text{ or}$$
$$w + iv = \frac{i}{x+iy}.$$

Separating into real and imaginary parts, we have

$$wx = vy, \text{ and}$$
$$vx + wy = 1.$$

These equations are identical with (6) and (7), if we interchange $w$ and $v$. Hence, it follows that bowing either part of the surface turned an angle of $\frac{\pi}{2}$ in the $xy$-plane, gives the other part of the surface.

The same thing can be shown geometrically by a consideration of the orthogonal systems

shown in figure 29. By turning the system of potential lines through an angle of $\frac{\pi}{2}$ about the origin, it is made to coincide with the system of stream lines. Since the stream lines of the real part of the surface coincides with the potential lines of the surface of imaginaries, it follows that a turning of the surface of reals through an angle of $\frac{\pi}{2}$ gives the surface of imaginaries.

This surface is number IV of Brill's models Series XIV.

We shall now take up the more general function,

$$(1) \qquad w = \frac{1}{2\varepsilon} \log \frac{z+\varepsilon}{z-\varepsilon},$$

which is represented by numbers Va and Vb of the Brill models. Placing the function in the exponential form, we obtain

$$(2) \qquad e^{2\varepsilon w} = \frac{z+\varepsilon}{z-\varepsilon}.$$

Making the following substitutions

(3) $$\frac{z+\varepsilon}{z-\varepsilon} = \rho(\cos\theta + i\sin\theta),$$

(4) $$e^{2\varepsilon w} = e^{2\varepsilon w + 2i\varepsilon v} = e^{2\varepsilon w}(\cos 2\varepsilon v + i\sin 2\varepsilon v),$$

we obtain

(5) $$e^{2\varepsilon w}(\cos 2\varepsilon v + i\sin 2\varepsilon v) = \rho(\cos\theta + i\sin\theta).$$

Equating moduli and amplitudes

(6) $$\rho = e^{2\varepsilon w}, \text{ and}$$

(7) $$\theta + 2k\pi = 2\varepsilon v,$$

in which $k$ is any positive integer.

From (6), we obtain

(8) $$w = \frac{1}{2\varepsilon}\log\rho,$$

which is the real part of the function. From (7),

(9) $$v = \frac{1}{2\varepsilon}(\theta + 2k\pi),$$

which defines the surface of imaginaries. Equation (9) is a <u>periodic function</u>, since for each set of values up to $2\pi$, which may be given to $\theta$, there corresponds an infinite number of other values of the function which are determined by adding some multiple of $2\pi$ to the original values of $\theta$. It can be seen that

function $v$) is represented by an infinite number of similar parts which are arranged at equal intervals. Each part corresponds to a particular value of $k$. For each value of $k$, there is obtained what is known as a <u>period of the function</u>. We shall discuss the function for the period in which $k$ is equal to 0. The same principles would hold for any other period. For this period, (9) becomes

$$v = \frac{\theta}{2\varepsilon} \tag{10}$$

Before going further we must evaluate $\rho$ and $\theta$ in terms of $x$ and $y$. From (3) by a substitution for simplification, we obtain

$$\rho(\cos\theta + i\sin\theta) = \frac{x+iy+\varepsilon}{x+iy-\varepsilon} = s+it, \tag{11}$$

in which $s$ and $t$ are functions of $x$ and $y$. By separating the real and the imaginary parts, we have

$$s = \rho\cos\theta, \text{ and}$$

$$t = \rho\sin\theta.$$

From these equations, we obtain

(12) $$\rho = \sqrt{s^2 + t^2}, \text{ and}$$

(13) $$\theta = \frac{1}{2\varepsilon} \operatorname{arctg} \frac{t}{s}.$$

From (11), we have

$$x + iy + \varepsilon = (x + iy - \varepsilon)(s + it).$$

Separating into real and imaginary parts and solving simultaneously for $s$ and $t$, we get

$$s = \frac{x^2 + y^2 - \varepsilon^2}{(x-\varepsilon)^2 + y^2}, \text{ and}$$

$$t = \frac{2\varepsilon y}{(x-\varepsilon)^2 + y^2}.$$

Substituting these values in (12) and (13), we have

$$\rho = \sqrt{\frac{(x+\varepsilon)^2 + y^2}{(x-\varepsilon)^2 + y^2}}, \text{ and}$$

$$\theta = \operatorname{arctg} \frac{-2\varepsilon y}{x^2 + y^2 - \varepsilon^2}.$$

Substituting these values for $\rho$ and $\theta$ in (8) and (10), we obtain

(14) $$w = \frac{1}{4\varepsilon} \log \frac{(x+\varepsilon)^2 + y^2}{(x-\varepsilon)^2 + y^2}, \text{ and}$$

115 $$v = \frac{1}{2\varepsilon} \operatorname{arctg} \frac{-2\varepsilon y}{x^2 + y^2 - \varepsilon^2}.$$

We shall now take up the discussion of these surfaces, giving to the constant $\varepsilon$ the value $\pi/4$. For the $w$-surface, we have the equation,

$$w = \frac{1}{\pi} \log \frac{(x+\frac{\pi}{4})^2+y^2}{(x-\frac{\pi}{4})^2+y^2}.$$

The traces which the surface makes on the coordinate planes can be found in a manner similar to that used in the preceding part of this section. However, since there are no branch- or cross-joints, the complete surface may be determined by giving to $w$ different constant values. We shall first put the equation in the exponential form, thus:

$$e^{\pi w} = \frac{(x+\frac{\pi}{4})^2+y^2}{(x-\frac{\pi}{4})^2+y^2}.$$

Clearing of fractions, we obtain

$$(1-e^{\pi w})(x^2+y^2) + (1+e^{\pi w})\tfrac{\pi}{2}x + (1-e^{\pi w})\tfrac{\pi^2}{16} = 0, \text{ or}$$

$$x^2+y^2 + \frac{1+e^{\pi w}}{1-e^{\pi w}} \cdot \frac{\pi}{2}x + \frac{\pi^2}{16} = 0.$$

It can easily be seen that in any plane which is determined for a particular value of $w$, the surface will cut out a circle of radius $\frac{\pi e^{\frac{\pi w}{2}}}{2(1-e^{\pi w})}$, and with centers at the point whose $x$ and $y$ coordinates are $-\frac{\pi}{4}\left(\frac{1+e^{\pi w}}{1-e^{\pi w}}\right)$ and 0. The centers of all of these circles, which are cut

cut by the surface will lie in the plane for which

$$y = 0,$$

and at a distance of $-\frac{\pi}{4}\left(\frac{1+e^{\pi w}}{1-e^{\pi w}}\right)$ from the w-axis. When w is equal to o this circle has an infinite radius. That is, the trace of the xy-plane on the surface is a straight line.

The equation for the v-surface when $\epsilon$ becomes $\frac{\pi}{4}$ is

$$v = \tfrac{2}{\pi} \operatorname{arctg} \frac{-\frac{\pi}{2} y}{x^2+y^2-\frac{\pi^2}{16}}, \text{ or}$$

$$\tan\left(\tfrac{\pi}{2} v\right) = \frac{-\frac{\pi}{2} y}{x^2+y^2-\frac{\pi^2}{16}}.$$

This can be put in the form,

$$\left\{\tan\left(\tfrac{\pi}{2} v\right)\right\}\left\{x^2+y^2\right\} - \tfrac{\pi^2}{16}\tan\left(\tfrac{\pi}{2} v\right) + \tfrac{\pi}{2} y = 0, \text{ or}$$

$$x^2+y^2+\frac{\frac{\pi}{2} y}{\tan\left(\frac{\pi}{2} v\right)} - \frac{\pi^2}{16} = 0.$$

Again, we have a system of circles cut out by the surface on the planes determined by the constant values of v. The centers of these circles lie at the points whose coordinates are, o and $-\frac{\frac{\pi}{4}}{\tan\left(\frac{\pi}{2} v\right)}$. The circle which lies in the plane

$$v = 0,$$

is of infinite radius or a straight line coincident with the x-axis. The other circles have their centers in the plane for which x equals 0. Since the sine is a periodic function, taking on equal values throughout each period of $2\pi$, the surface of imaginaries will have an infinite number of similar parts. Since the periodicity depends on the values of $\sin \frac{\pi}{2} v$, it follows that a series of horizontal planes which cut the v-axis at points which are 4 units apart, will separate the periods of the surface. The first period, that is the one containing the origin, will be defined between the planes which cut the w-axis at $\pm 2$. Figure 30 shows the projection of the orthogonal systems of curves on a horizontal plane. The surface of reals does not differ essentially from the surface of reals for the function,

$w = \frac{1}{z}$.

The surface of imaginaries, however, will call for special attention. The projection of the lines,

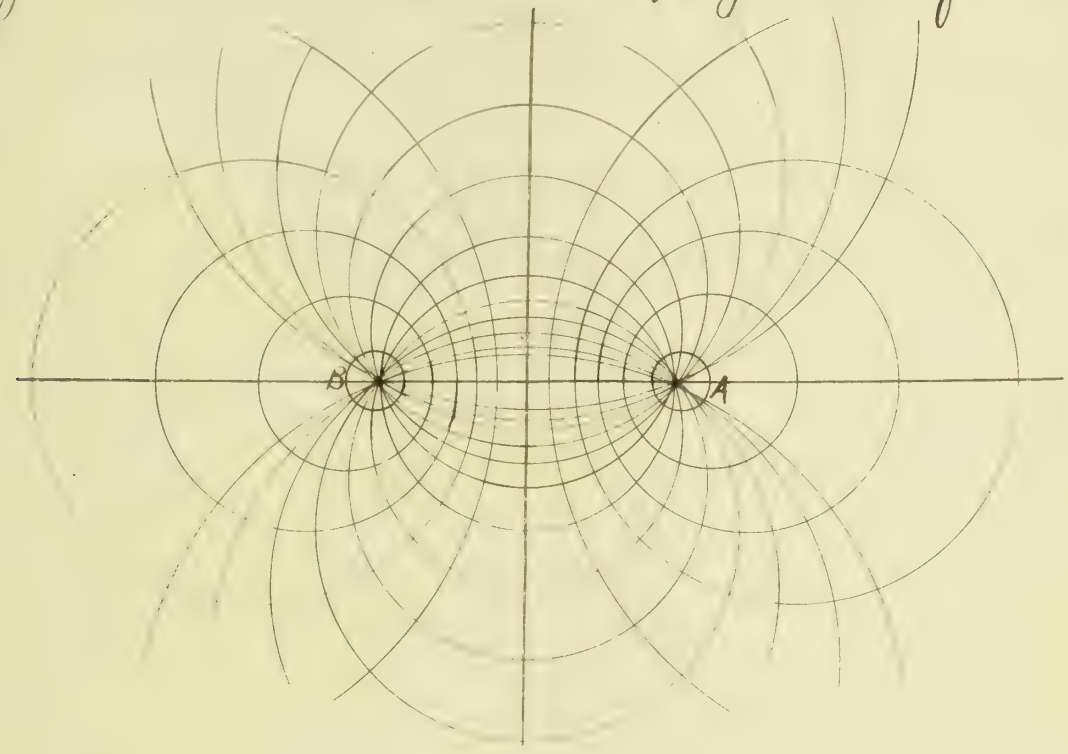


Fig 30

cut out by the horizontal planes from the the surface of imaginaries, gives a system of circles which pass through A and B (see figure 30). We can locate the points A and B by finding where the circle, which lies in a plane defined by a particular constant value of v, pierces the $wx$-plane. Take the horizontal plane whose equation is

$v = 1$.

The equation of the circle in this plane which may be obtained by substituting 1 in the general equation for the system of circles, is

$$x^2 + y^2 + \frac{\frac{\pi}{2} y}{\tan \frac{\pi}{2}} - \frac{\pi^2}{16} = 0.$$

Since $\tan \frac{\pi}{2}$ is $\infty$, the equation reduces to

$$x^2 + y^2 = \frac{\pi^2}{16}.$$

Solving this circle simultaneously with the equation

$$y = 0,$$

we obtain

$$x = \pm \frac{\pi}{4}.$$

Consequently, if we suppose that all of the circles are projected on the xy- coordinate plane, the points A and B will lie on the x-axis at a distance of $\frac{\pi}{4}$ from the origin. It follows then, that in the surface, the circles cut out by horizontal planes will all pass through the lines, which lie in the wx-plane, and whose equations are,

$$x = \pm \frac{\pi}{4},$$

or going back to the original form of the function, the lines for which

$$x = \pm \varepsilon,$$

It follows from section 16 that the circles in the horizontal planes, lying infinitely near each other, will make up the completed surface. It will be noticed that in the surface of imaginaries, there will be a strip lying in the $ux$ plane bounded by the lines,

$$x = \pm \frac{\pi}{4},$$

and extending out to infinity, which contains no part of these horizontal circles. The surface is discontinuous for this strip. However, those circles for which the radius, $\frac{\pi}{4} \frac{1}{\sin(\frac{\pi}{2} v)}$, becomes infinite, that is become straight lines, will cross this discontinuous strip. This occurs when $v$ equals 0 or some multiple of 4. These lines mark the different periods of the function.

23. Function $6w = e^{\frac{1}{6z}}$. We now take up the discussion of a function whose surface contains a point of essential singularity. The function

(1) $$6w = e^{\frac{1}{6z}},$$

which is represented by Brill's models, Number VI. Separating the function into its real and imaginary parts, we have

(2) $$\begin{aligned} 6(w+iv) &= e^{\frac{1}{6(x+iy)}}, \\ &= e^{\frac{x}{6(x^2+y^2)} - \frac{iy}{6(x^2+y^2)}}, \\ &= e^{\frac{x}{6(x^2+y^2)}}\left\{\cos\frac{-y}{6(x^2+y^2)} + i\sin\frac{-y}{6(x^2+y^2)}\right. \end{aligned}$$

From this, it follows that

(3) $$w = \frac{1}{6}e^{x'}\cos y', \text{ and}$$

(4) $$v = \frac{1}{6}e^{x'}\sin y',$$

in which

(5) $$x' = \frac{x}{6(x^2+y^2)}, \text{ and}$$

(6) $$y' = \frac{-y}{6(x^2+y^2)}.$$

If we go back to the original function and transform it by reciprocation in the xy-plane,

we obtain

(7) $$6w = e^{\frac{z'}{6}}.$$

Separating this into its real and imaginary parts, we have

(8) $$6(w+iv) = e^{\frac{x'+iy'}{6}},$$

$$= e^{\frac{x'}{6}}\left(\cos\tfrac{y'}{6} + i\sin\tfrac{y'}{6}\right).$$

From this, we obtain

(9) $$w = \tfrac{1}{6}\, e^{\frac{x'}{6}} \cos\tfrac{y'}{6}, \text{ and}$$

(10) $$v = \tfrac{1}{6}\, e^{\frac{x'}{6}} \sin\tfrac{y'}{6}.$$

It will be noticed that equations (9) and (10) differ from (3) and (4) by only a constant factor. It follows, then, that for all theoretical results, (3) and (4) may be considered as the reciprocals of the original functions. These equations are not only simpler for computing, but they also bring the point $\infty$ over into the point 0, which is especially desirable. It enables us to find the nature of the region at infinity by studying its reciprocal around the origin. The

behavior of the transformed function in the neighborhood of the origin is the same as the behavior of the original function at infinity.

By uniting (3) and (4), we have

$$w = w + iv = \frac{1}{6} e^{x'}(\cos y' + i \sin y').$$

If we reflect the function with respect to the w axis, we change the amplitude y' into an amplitude of equal magnitude but of opposite sign. The angle then becomes $(-y')$[x]. We then have

$$w = \frac{1}{6} e^{x'}\{\cos(-y') + i \sin(-y')\},$$

which can be put in the form

$$w = \frac{1}{6} e^{x'}(\cos y' - i \sin y').$$

If we now turn the w-plane through an angle of $\frac{\pi}{2}$, or do what will be the same thing, multiply the right hand member of the Equation by i, we obtain

$$w = \frac{1}{6} e^{x'}(\sin y' + i \cos y').$$

Separating this transformed w into its real and

---

(x) Burkhardt, Analytischen Functionen, section 11.

imaginary parts, we have

$$u = t e^{x'} \sin y', \text{ and}$$
$$v = t e^{x'} \cos y'.$$

By comparing these equations which have been obtained by a reflection and a turning through an angle of 90° with the original equations, we see that $u$ and $v$ have changed places. It follows, then, that the transformations which have just been performed change the surface of reals into the surface of imaginaries, and conversely. Moreover, since the transformations do not bring in any new essential characteristics of the function, we need only to consider the real part of the function.

By putting the equation for the surface of reals,

$$u = e^{x'} \cos y',$$

in the logarithmic form, we have

$$\log w - \log \cos y' = x'.$$

We are now able to plot the system of equal potential lines which correspond to successive values of w. It will be noticed that when z becomes zero, that is x and y each becomes zero, the function has the form,

$$\log w = 0.$$

Therefor, each equal potential line passes through the w-axis regardless of the value of w. An infinite number of these lines will pass through the zero point, when the curves are projected on one of the horizontal planes Figure 31 shows the projections of the orthogonal systems on one of the horizontal planes. It will be noticed that each curve of both systems passes through the point o. An infinite number of curves pass through the point, consequently it is a cross point of infinitely high order, or a <u>point of essential singularity</u>.

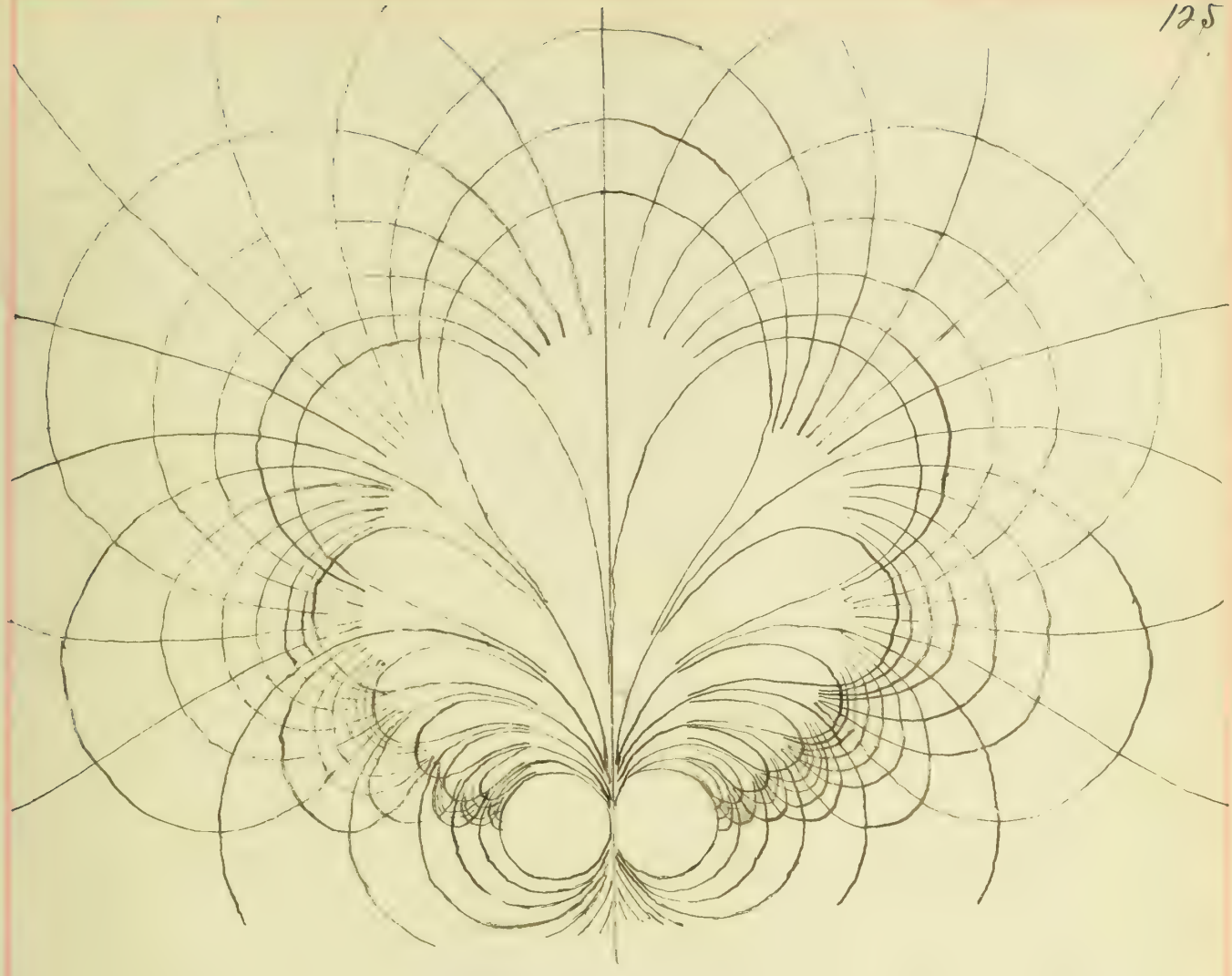

Fig 31

In the preparation of this Thesis the following authorities have been examined:

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Klein, On Riemann Theory of Functions.

Clifford, Mathematical Papers.

Forsyth, Theory of Functions.

Durège, Elements of the Theory of Functions.

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Burnside and Panton, Theory of Equations.

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